

Complex numbers are basis rotations: $\hat{k} \times$ and the $\hat{k} \times$ basis exponential

I have gone out prospecting (practically every day) for new mathematical equations for the good part of 20 years. I have now got it to a point where I discover mathematical gold practically every time I go out prospecting; the **log-normal theory of human life-span** was the first time I struck gold - human lifespan is Gauss distributed as the logarithm of age from the species maximum (as are probably all eukaryotic organisms, the maximum determined by chromosomes of a particular species). I have ended up producing more theory than I had originally anticipated with this. It has taken me just over four months. I had originally thought it too trivial ($i = \hat{k} \times$) and was only going to include it in future research as a tool. Now it is quite apparent that it is not trivial. So here it is:

What are complex numbers?

One of the things I do is develop new maths and new mathematical equations. I have found this to be my best trait (after doing the research degrees, **actuarial honours** and **finance PhD**). A very significant proportion of all useful maths involves complex numbers and when I have wanted and tried to extend for new equations (which I have been pretty successful at in the past for real quantities despite receiving no credit for it) that involve complex numbers I have run into difficulties. Two notable cases where this occurred involving complex numbers was when **(1)** I was extending my document on power series to Laurent series and the Cauchy integral formula (which I was extending for generalizations) and **(2)** at earlier times when I was doing all the maths for the PhD Supplement involving harmonic Brownian motion [which I managed to get everything to real scalars]. These difficulties mostly involved having to ask myself “what actually is a complex number?”. I think many people involved in mathematical research have asked themselves the same question, the standard answer to this question being “the square root of negative one” not being sufficient nor entirely satisfactory. Probably ever since the invent of the idea of complex numbers by the Italians almost five centuries ago. I think there have even been philosophical arguments over “what a complex number actually is” [... which I was trying to hold out on this for literary effect... it is a basis curl, that is, a basis cross product $i = \hat{k} \times$; k -cross, short for k -hat basis cross product].

Plane geometry and algebra is the mathematics handed down to us and this mathematics is scalar. So we, as mathematicians, have a preoccupation with scalar functions when actually everything we are trying to model occurs in a three-dimensional vector space, that of a vector volume. Complex numbers have been our solution to keeping things scalar (-like) despite the fact that everything occurs vectorially in R^3 .

It is now my contention that there is nothing complex about a complex number nor quaternion. Nor are they numbers. These are rotational manipulations of $\hat{i}$, $\hat{j}$, and $\hat{k}$.

This is the reason for this document. It has become increasingly apparent to me, after taking several engineering courses and then taking complex analysis formally [MATH2521 Complex Analysis], whilst solving Rubik's cubes and just after sitting MATH2011 vector calculus (my background in first undergraduate was actuarial and statistical, it was undergraduate again because the finance PhD examiner said I had to start again) and now just after sitting some more linear algebra MAT2501] that complex numbers are not an extended field of real numbers (i.e., not an extension of reals R to some additional field C) but are basis vectors, in particular, cross products of the standard basis vectors $\hat{i}$, $\hat{j}$, and $\hat{k}$; specifically, the standard basis vectors appended with $\times$. All of the mathematics that involves these, then, is entirely real (on R) in a 3-dimensional vector volume (R^3).

In civil engineering (MMAN1300 Engineering Mechanics [2017]) I became convinced that $\hat{i}$, $\hat{j}$, and $\hat{k}$ are complex numbers given that all outcomes are always real despite being overwhelmingly concerned with rotations rotational inertia and torque (moments coupled with linear forces). Consequently, I had quite obviously tried squaring them which quite obviously didn't lead anywhere.... why? because bases are vectors and you can't square a vector... you can only take outer product, inner product, dot product, and cross product... turns out that cross product does the trick... didn't realize this straight away... several years elapsed from 2017 (and sitting and ENNG2300 Mechanics of Solids [2019] and MATH2521 Complex Analysis [2019]) and it was Rubiks cubes that I finally made the obvious connection with $\times$ and the standard basis with complex $i = \sqrt{-1}$ in 2020... since it was not my first degree, I could fully appreciate what these courses were teaching: I was at the time, and still, searching for a theory of everything in terms of calculus... the question "what actually is a complex number?" being probably of the highest importance and of equal importance now is to recognize that it is not a number (as such). So from becoming convinced that $\hat{i}$, $\hat{j}$, and $\hat{k}$ are complex numbers, it has turned out to be correct once appended with $\times$ which convert these variables to rotational operators with the exact properties of complex numbers, most notably that of $\sqrt{-1}$.

So I essentially became certain of the fact that complex numbers are rotational basis operators in engineering [courses and studying Miriam and Kraige], in which all productive outcomes are real and almost always coupled with torque and shear, and with my major interest in, and ongoing experience with, **Rubik's cubes** which I had begun to think of as a **self-contained system of complex numbers** and subsequently adapted cubing algorithm notation to correspond with complex theory. This adapted notation is not perfect although I have found it to be an excellent first step for a full mathematical theory of the Rubiks cube. Rotations in each slice of a Rubik's cube individually work exactly as an Argand diagram in how we understand the workings of complex numbers as the **quartic roots of unity**. R is the cubing notation for a right spin, that is, a quarter rotation (I use the notation i for this). So, for instance, R, R², R³, R⁴ returns a right handed rotation back to the starting position similarly as 4 multiplications by i , that is, i , i^2 , i^3 , i^4 (spinning around i -hat) without having to enter into any 'complex plane' or some extended space from the reals R. This is similarly with rotation in any slice, F or U for standard cubing notation (for which I use j and k). For example, rotating around the x axis as above with R or rotating around the y or z axes with F or U [I use a left handed complex notation for my cubing with lev (laevorotary) and dex (dextrorotary) for left and right hand moves]. This is similar to making rotations with eigenvalues and eigenvectors for which rotations can be made without any reference to complex numbers, only sines and cosines in a linear algebra basis. The equivalent for top slice (layer) rotations is U, U², U³, U⁴ which returns the cube back to the starting position before the first U move. The notation that I have started to use for this (about two years ago) was k , k^2 , k^3 , k^4 (rotating around k -hat) which similarly returns it to the starting position, equivalent to the identity I (four quarter turns of a rotation matrix). With this notation I am sure that most who have worked with complex numbers and basis vectors could see how I would start thinking that complex numbers are essentially basis variables (with the cubing algorithm notation adapted for complex theory). Similarly for front moves F (which I use the notation j), for which F, F², F³, F⁴ returns the cube back to the starting position before the first F move. In the notation I use and learn algorithms with (written by cube solvers and writers as R, F, and U) this is j , j^2 , j^3 , j^4 (for F, F², F³, F⁴) which rotates in discrete quarter rotations about j -hat. Again, these moves move the cube back to the starting position similarly as the mathematics for complex numbers on the Argand diagram but clearly without the need for any extension into any imaginary complex field. A system of three orthogonal Argand diagrams, for example, is not required nor any extension into any quaternion field as it is all real in the R^3 vector space.

The reason for this, that no extension to any complex field is required, I am now certain. All mathematics (measurable variables and manipulation of those variables) occurs in 3-space.

This space is generated mathematically by the basis vectors $\hat{i}$, $\hat{j}$, and $\hat{k}$

The algebra that we all know and love, including the geometry passed down from Euclid, is essentially plane algebra, that is, the mathematics of plane surfaces. This space is generated by any two of the three basis vectors such as $\hat{i}$ and $\hat{j}$. So here is the crux of the issue:

The basis vectors $\hat{i}$ and $\hat{j}$, for the plane, do not have an independent existence from $\hat{k}$, that is, a plane is always associated with the remaining dimension for a volume (similarly as a Rubik's cube).

In plane algebra this dimension that completes the span for the basis into a volume ($\hat{k}$) is latent (hidden).

The extraneous basis variable ($\hat{k}$) is then introduced into the mathematics by what we understand as the complex number $i = \sqrt{-1}$ when it is found that the algebra requires this (defining $i = \sqrt{-1}$) for a valid solution. The (crude algebraic) definition $i = \sqrt{-1}$ is the extension of the plane onto the latent axis and movement into the 3rd dimension of a real R^3 volume. This was usually a negative quadratic determinant as first recognized and introduced into mathematics by the Italians 500 years ago pre-Galileo. Further requirement for this was then the overwhelmingly important extension of complex numbers in the theory of quadratic equations to the Fundamental Theorem of Algebra by Gauss some 200 years ago, and the need to introduce (slightly earlier by Euler) the notation $i = \sqrt{-1}$ for higher order equations (that is, equations of arbitrary and arbitrarily large order) and rotational Euler equations. This also applies to plane curves. All curves on an R^2 plane are actually particular cases of curves in an R^3 volume, which a rotation matrix around the x-axis or the y-axis demonstrates (by giving the curve a $\hat{k}$ component). Similarly, any curve in a volume can be brought onto a plane for plane mathematics by application of $\hat{k}$ -cross (albeit with the loss of the data on the $\hat{k}$ -hat component).

This means that the Argand diagram (Gaussian plane) is actually the real x-y plane when i is considered in this way, as the **curl** of [rotation around] $\hat{k}$ -hat, that is, the basis variable $\hat{k}$ -hat appended with $\times$ [note that it is called a product but it is a rotation operator].

Also, as shown later, scalar functions can be placed on any of the three basis variables ($\hat{i}$, $\hat{j}$, $\hat{k}$) in the three dimensional vector space, but those functions (operators) involving $\hat{k}$ -cross, $\hat{j}$ -cross, and $\hat{i}$ -cross are mapped to the three basis variables in R^3 in specific ways, and once this mapping is made, the functions (operators) convert to scalars. It means that this mapping is not arbitrary for the conversion of an operator to a scalar upon mapping to a basis variable.

I made this deduction, that complex numbers are basis operators (not numbers) and not complex or imaginary but better described as a **latent axis operator**, based pretty much entirely by reference to Rubik's cubes [and the courses I was sitting] because I then knew that everything with the rotations is real in R^3 [so then I just had to deduce how]. Erno Rubik's invention deserves a Nobel Prize in Physics for this deduction alone [independently of the many more to come as a result of the brilliant physical invention]. I have actually tried (by email) to see if the Swedish would award **Erno Rubik** with a **Nobel Prize in Physics** for his invention because of the future breakthroughs I know it will allow quantum mechanics (some of which probably by me, there are probably quite a few in here already actually, which I worked out whilst writing it). If they don't, it will be a missed opportunity for them.

This will essentially mean that all of complex analysis and calculus can be converted into real calculus in the vector volume space generated by $\hat{i}$, $\hat{j}$, and $\hat{k}$. This will make this mathematics much more useful for applications, and will allow for many further extensions by correlating the 3 complex fields [now known to not be complex but rather the three real mutually orthogonal R^2 planes

that comprise R^3 with the three **quarter rotation, single component basis cross products** $\hat{i} \times$, $\hat{j} \times$, and $\hat{k} \times$] into one another since $\hat{i}$, $\hat{j}$, and $\hat{k}$ can interact, whilst keeping all the mathematics real, i.e., all on R (the reals) on the 3 axes generated and spanned by $\hat{i}$, $\hat{j}$, and $\hat{k}$.

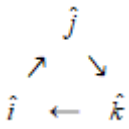
So, like all good mathematics, it is very simple $i = \hat{k} \times$

[I was trying to hold out on defining it until here]

It seems simple, and it is simple, yet is as of yet to be realized and stated. This is the purpose of this writing. There are not many higher order dimensions. There are three and only three. These are x, y, and z; left-right, forward-backward, and up-down. If only two are stated [plane algebra], one is latent. If only one is stated [scalar], then two are latent.

This will allow us to keep all of the mathematics real (in R^3) and hence solve problems that I have been having problems with regarding ‘what a complex number actually is’, **the definition that it is the square root of negative one being insufficient for my purposes** [hence the introduction paragraph]. Remaining impediments to extending the mathematics will then be those in the real vector space R^3 , which is much more intuitive and hence can provide for more confident applications when breakthroughs (new equations) are made. I am hoping that Rubiks cubes and this mathematics will allow us to mathematically waveshape arbitrarily sophisticated magnetic fields. I have become very interested in magnetic vector potentials, and these are curled vectors.

Here is how the theory works: it is the cross-product triangle, that is, the basis vectors $\hat{i}$, $\hat{j}$, and $\hat{k}$ appended with cross $\times$. Hence, it works exactly as the cross-product triangle and, moreover, whatever that can be derived from the cross product triangle is applicable to the theory of $i = \hat{k} \times$ for that axis. I have focused on k -cross because this relates to the x-y plane and rotations on that plane (and hence corresponds to what was thought of as complex i). Remembering for a right handed system that



so

$\hat{i} \times \hat{j} = +1\hat{k}$	$\hat{j} \times \hat{k} = +1\hat{i}$	$\hat{k} \times \hat{i} = +1\hat{j}$	$\hat{i} \times \hat{i} = 0\hat{i}$
$\hat{j} \times \hat{i} = -1\hat{k}$	$\hat{k} \times \hat{j} = -1\hat{i}$	$\hat{i} \times \hat{k} = -1\hat{j}$	$\hat{j} \times \hat{j} = 0\hat{j}$
			$\hat{k} \times \hat{k} = 0\hat{k}$

which are cyclic permutations of the 3 Cartesian axes, so that the components of the cross-product triangle of the standard basis are complex numbers [rotational operators] in R^3 as it is $\times$ appended to the standard basis which is generating the cycle. Note further than the cross-product triangle is overwhelmingly important to civil and mechanical engineering [and soon quantum mechanics] which is definitely not a coincidence that this correlates with the importance placed on the complex number $i = \sqrt{-1}$. Again, the components of the R^3 basis $\hat{i}$, $\hat{j}$, and $\hat{k}$ are the complex numbers when placed in the context of cross ($\times$) which converts the basis components to rotational operators with the exact properties of $i = \sqrt{-1}$.

If you are working on the x-y plane, which is generated by $\hat{i}$ and $\hat{j}$, then the latent axis will be in the z direction. The basis variable for this axis is $\hat{k}$ and consequently what we know as a complex number i is the (discrete) permutation **curl** on (quarter rotation around) this axis $i = \hat{k} \times$ (k -hat cross product) where it must be understood [for mathematical competence] that this is a (rotation) operator $\hat{k} \times (\cdot)$ that operates (always) on the other basis vectors $\hat{i}$ and $\hat{j}$ (and only on $\hat{i}$ and $\hat{j}$) and does not have an independent

existence from $\hat{i}$ and $\hat{j}$. Hence, $\hat{i}$ and $\hat{j}$ should be explicit to avoid difficulties or unspecified directions (if a basis variable is not explicit then it is latent but still there).

Similarly, if you are working on the x - z plane, which is generated by $\hat{i}$ and $\hat{k}$, then the latent axis will be the y axis. The basis variable for this axis, that which generates the space, is $\hat{j}$ and consequently what we know as a complex number i for this space will be $i = \hat{j} \times$ ($\hat{j}$ -hat cross product) which is similarly a (rotation) operator $\hat{j} \times (\cdot)$ for the **curl** around y and operates on $\hat{i}$ and $\hat{k}$.

Noting that I am using the term **curl** here to mean a **quarter circle rotation** about an axis generated by the **latent basis vector**, in other words, the single basis component cross product. Since we are considering plane algebra and, as I am hypothesizing here, all mathematics actually occurs vectorially in real three space R^3 , there are 3 dimensions from which 2 axes are chosen for a plane (${}^3C_2 = 3$) for which one is always remainder and hence latent for any plane mathematics (algebra or geometry). Hence, there are 3 such ways to choose such a combination of a plane and latent axis, and hence there are three different complex numbers.

Continuing for the third and last combination; if you are working on the y - z plane, which is generated by $\hat{j}$ and $\hat{k}$, then the latent axis is the x axis which corresponds to the latent basis variable $\hat{i}$. What we know as a complex number i for this space is then the discrete **curl** about the latent x axis $i = \hat{i} \times$ ($\hat{i}$ -hat cross product) where this is a (rotation) operator $\hat{i} \times (\cdot)$ and operates on the other basis vectors $\hat{j}$ and $\hat{k}$ which are implied (latent) if not stated explicitly for R^3 .

The complex number for any volume is then the basis cross products (single component cross products), $\hat{k}$ -cross, $\hat{j}$ -cross, and $\hat{i}$ -cross, $\hat{k} \times (\cdot)$, $\hat{j} \times (\cdot)$, and $\hat{i} \times (\cdot)$, for the planes spanned by $(\hat{i}$ and $\hat{j})$, $(\hat{i}$ and $\hat{k})$, and $(\hat{j}$ and $\hat{k})$ respectively. These operators I abbreviate to $\hat{k} \times$, $\hat{j} \times$, and $\hat{i} \times$ from $\hat{k} \times (\cdot)$, $\hat{j} \times (\cdot)$, and $\hat{i} \times (\cdot)$. This essentially renders the complex field redundant, as it is exactly equivalent to the R^2 vector space for any orthogonal plane, although an outstanding concept that has served well for hundreds of years. It is remarkable that it works at all given that it is purely algebraic. I still revert to the standard complex notation for convenience [and sitting exams] and for Rubiks cubes.

For serious calculation and resolving for mathematical research [especially for spatial systems], the full notation is necessary, that is, for a full elucidation with a full specification of all quantities in R^3 . This has relevance to electrical engineering, electromagnetism, and quantum mechanics, as well as 3D CAD systems [for mechanical components and civil structures] where basis systems are overwhelmingly important (all 3D modelling is based on the R^3 basis).

In simplest form, it really just means writing $\hat{k} \times$ instead of i , and then working with vectors [subject to quite a few nuances, mostly regarding commutivity]. It is literally possible to just replace i wherever it occurs with $\hat{k}$ -cross (or $\hat{j}$ -cross or $\hat{i}$ -cross depending on the circumstance, that is, depending on which plane is the working plane) and assigning a suitable direction (basis-hat) to the components, a zero phase basis for instance ($\hat{i}$ -hat to the x -component for a zero phase angle).

The recognition of this, and the conversion of all of the complex analysis and calculus to the reals R into the volumetric vector space R^3 generated by $\hat{i}$, $\hat{j}$, and $\hat{k}$ will then be a major step forward in our understanding of mathematics and physics. This brings together complex theory, vector calculus, and linear algebra which are the core of 2nd year and all higher applied mathematics.

All of the same properties hold, for differentiating and integrating complex numbers and exponentials and Euler's theorem [and much more, again subject to quite a few nuances such as invertibility], but it is made explicit what is going on and 'how and why the mathematics works'. This is a major preoccupation with many of us whom seek mathematical knowledge and understanding, and hence competence in application of theory.

$\hat{k} \times$, $\hat{j} \times$, and $\hat{i} \times$ are left to right operators: multiplied from the left, operate to the right (and resolve right to left), and are indifferent to scalars and functions other than functions of the other **basis variables** which **generate** and hence **label** and **placemark** the scalars and functions of relevance in space (in the volumetric space of interest).

To demonstrate, I'll provide an example now. All quantities (scalars and functions) are to be associated with a direction, that is, with one of the basis variables indicating its direction. In other words, all quantities are assigned a position in a valid 3-dimensional vector space (generated by $\hat{k}$, $\hat{j}$, and $\hat{i}$).

Algebra

As an example, say you are wanting to find the roots of $y = x^2 + a^2$ so you are working on x-y the plane. You would zero the y and solve

$$0 = x^2 + a^2$$

$$x^2 = -a^2$$

$$x = \pm\sqrt{-a^2} = \pm a\sqrt{-1} = \pm ai \quad \text{where } i = \sqrt{-1}$$

Here I will show that $i = \sqrt{-1}$ is equivalent to $\hat{k} \times$. Let's say that y is in the usual direction for $\hat{j}$ (a zero phase basis). Then

$$y \hat{j} = (x^2 + a^2) \hat{j}$$

zero the y for the root

$$(0) \hat{j} = (x^2 + a^2) \hat{j}$$

$$x^2 \hat{j} = -a^2 \hat{j}$$

and left to right multiply both sides through by the squared basis **curl** $\hat{k} \times$ (which is the operator corresponding to the complex number, here, $i^2 = -1$), i.e., $(\hat{k} \times)^2$

$$(\hat{k} \times)^2 x^2 \hat{j} = (\hat{k} \times)^2 (-a^2 \hat{j})$$

$$x^2 (\hat{k} \times)^2 \hat{j} = (-a^2 (\hat{k} \times)^2 \hat{j})$$

scalars indifferent to k-cross and hence commute

$$x^2 (\hat{k} \times) (\hat{k} \times) \hat{j} = -(a (\hat{k} \times))^2 \hat{j}$$

and then with k -cross squared applied to j -hat

$$(\hat{k} \times)^2 \hat{j} = (\hat{k} \times)(\hat{k} \times) \hat{j} = (\hat{k} \times)(\hat{k} \times \hat{j}) = (\hat{k} \times)(-\hat{i}) = -(\hat{k} \times \hat{i}) = -(\hat{j}) = -\hat{j} \quad \text{i.e., } (\hat{k} \times)^2 \hat{j} = (-1) \hat{j}$$

the equation becomes

$$\begin{aligned} -x^2 \hat{j} &= -(a(\hat{k} \times))^2 \hat{j} \\ x^2 \hat{j} &= (a(\hat{k} \times))^2 \hat{j} \end{aligned}$$

which both are on the same y axis and of the same sign so are scalar

$$x^2 = (a(\hat{k} \times))^2$$

(alternatively obtained by right to left dotting both sides by $\hat{j}$). Taking the square root

$$x = \pm a \hat{k} \times$$

which is the solution in terms of the **basis curl [rotational operator, k -hat cross product, k -cross]**. Comparing with the complex number notation $x = \pm a i$ it is clear that these two are equal when $i = \hat{k} \times$. So if the solution requires that x be in the $\hat{i}$ direction [the zero phase even direction], then the solution is

$$x \hat{i} = \pm a \hat{k} \times \hat{i} = \pm a \hat{j}$$

Noting that the axis in the j -hat direction (the j -hat component) is the real y axis, not an imaginary non-existent axis. Note further, at this early stage, that k -cross (complex number) and the k -hat exponential (complex exponential) at $\pi/2$ are not equal, despite what Euler's formula suggests (although they appear to be when k -hat is ignored, that is latent, when working in scalars on the complex plane [which is actually R^2]). This, $x \hat{i} = \pm a \hat{j}$, can be returned back to the earlier form, $x = \pm a \hat{k} \times$, by multiplying both sides by $\hat{k} \times$ and cancelling $\hat{j}$ on both sides or alternatively dotting both sides with $\hat{j}$

$$\begin{aligned} x \hat{i} &= \pm a \hat{j} \\ x \hat{k} \times \hat{i} &= \pm a \hat{k} \times \hat{j} \\ x \hat{j} &= \pm (a \hat{k} \times) \hat{j} \\ x \hat{j} \cdot \hat{j} &= \pm (a \hat{k} \times) \hat{j} \cdot \hat{j} \\ x &= \pm a \hat{k} \times \end{aligned}$$

since $\hat{j} \cdot \hat{j} = 1$

Hence $i = \hat{k} \times$ is definitively equivalent to the complex number $i = \sqrt{-1}$ in that it has exactly the same properties that made it necessary to define the complex numbers when the concept of $\sqrt{-1}$ was first introduced, notably, a negative quadratic determinant for solutions to quadratic equations as per the example above.

This has implications for the **Fundamental Theorem of Algebra**. The Theorem states that a polynomial of degree n can be factored into n roots, which are either complex or real, and all complex roots occur as conjugate pairs. The consequence, then, is that a factored quadratic pair of roots is not fully specified if it is not assigned a direction and stated with which basis **curl** [which single component basis cross product] it is associated, since there are three types rather than one. So for the factorization to be fully resolved, it needs to be explicitly stated whether the conjugate pair is in $\hat{k} \times$, $\hat{j} \times$, or $\hat{i} \times$ as per the demonstrated equivalence of the complex plane to a real R^2 vector space in terms of basis rotations from basis cross products in the real Cartesian volume R^3 . Note importantly also that roots which are functions of k -cross, j -cross, and/or i -cross do not commute as scalar functions. These are **non-commutative** operators, not numbers, and are indifferent to scalars.

I will be showing later in this document that these relate to diffusion equations for Rubiks cubes, although this is deferred for the time being because of work here developing the theory of the k -hat exponential **expk** and Riemann sphere which are discussed shortly (where I define **expk**, **sink**, **cosk**, as well as **expj**, **sinj**, **cosj**, and **expi**, **sini**, **cosi**, as well as how and why I defined them). Rubiks cubes algorithms are discrete diffusion equations of arbitrary order and scrambles are random walks; random walks of curl which might be best called rotational random walks (quantized random walks without divergence).

For a Rubiks cube, the right hand slice rotations R, R^2, R^3, R^4 returns the right handed rotation back to the starting position similarly as 4 multiplications by i , that is, i, i^2, i^3, i^4 , which are, again, actually $\hat{i} \times, (\hat{i} \times)^2, (\hat{i} \times)^3, (\hat{i} \times)^4 = (\hat{i} \times)^0 = 1$ as a spin [a sequence of quartic rotations] around i -hat exactly one-for-one with a Rubiks cube [ignoring the rotation eigenvalue and eigenvector]. Similarly with U, U^2, U^3, U^4 and F, F^2, F^3, F^4 which are similarly as k, k^2, k^3, k^4 and j, j^2, j^3, j^4 which are actuarially one-for-one with $\hat{k} \times, (\hat{k} \times)^2, (\hat{k} \times)^3, (\hat{k} \times)^4 = (\hat{k} \times)^0 = 1$ and $\hat{j} \times, (\hat{j} \times)^2, (\hat{j} \times)^3, (\hat{j} \times)^4 = (\hat{j} \times)^0 = 1$. All Rubiks cubes algorithms carry over one-for-one to the theory of basis **curls** (basis rotations) as basis cross products (which supersedes, as basis rotation operators, scalar imaginary numbers). Hence this makes possible the most interesting application as an extension and move beyond complex numbers, that of interacting the operators. Without interacting the rotations, the interaction between the 3^3-1 (mechanism) = 26 bits on the three axes (the mechanism) of a Rubiks cube would be boring. As more interesting that they are because of this is how much better and more interesting mathematics is going to get. That is, this allows for the possibility of interacting the spins on the 3 different axes. The Rubik's cube essentially tells us how to do it, that is, interact the 3 different axes with what we have known as complex numbers.

Just as I became convinced that complex numbers are basis variables [i -hat, j -hat, and k -hat, which is so once appended with $\times$], I also become convinced that Rubik's cubes [which I also became convinced are systems of complex numbers] are essentially the elements of a rectangular coordinate system, that of R^3 since a coordinate system is that defined by its basis [i -hat, j -hat, and k -hat].

The 6 centers [faces] are the 3 coordinate axes, one face a positive pole and the opposing face the negative pole, of the x axis, y axis and z axis. The twelve ($12 = 3*4$) edges with the 6 centers [faces] form the middle slices which are the three mutually perpendicular Cartesian planes, the x - y plane, the y - z plane, and the z - x plane. The 12 edges are the 4 quadrants of the 3 three Cartesian planes. The corners are the eight octants of R^3 , that is, of the Cartesian volume of 3-space. The mechanism is the vertex of the system, known in

mathematics as the origin and the intersection of the three mutually perpendicular axes [generated by i -hat, j -hat, and k -hat] around which the quadrants [edges] and octants [corners] spin and permute. This is what you see when you take a Rubiks cube apart (i.e., the mechanism, which is an R^3 basis system of 3 mutually perpendicular axes exactly as for the axes of mathematical Cartesian 3-space).

Also, the equations follow the rules of transposes and inverses of orthogonal matrices $(AB)^T = B^T A^T$ where $A^T = A^{-1}$ which is the rule for the definition of an orthogonal matrix. This means that all of the Rubik's cubes algorithms are reversible, as all cube solvers know, so that any algorithm can always be performed in reverse and this corresponds to transposing all of the operators (R, U, F, and so on) and reversing the order from the start of algorithm to the end of the algorithm. This is exactly the matrix algebra of orthogonal matrices, one-for-one and hence the indicator to me of the redundancy of the complex field and a much much richer mathematics being as of yet unknown [but now within reach][although **complex numbers** have been a necessary idea that has served very well and made these deductions possible and can now, where appropriate, be converted to real vectors in R^3 ... still obviously super cool, I am just making the point, and it is definitely a way to move mathematics forward]. I envisage that every complex formula can be converted to reals in an R^3 vector volume which will make these formulae immensely more useful for real physical applications and outcomes. No need to throw away the imaginary part for example.

If you think about a dynamo that produces electricity for a city or town's grid, the turbine is always spinning on a single axis. For Rubik's cubes, there are situations where single-component-basis-cross-products are possible whilst arbitrary vector-cross-products are not. Specifying them as arbitrary vector cross products makes the maths overly difficult if not impossible, for example, if there is only one degree of rotational freedom when it would otherwise be simple with a single basis cross product. In which case it then has the properties and simplifications of a complex number. Turbines and spinning shafts also almost always transfer energy (magnitude and direction) to other shafts with cogs and gears. These are also single-component-basis-cross product spins, rather than arbitrary vector cross products [unless also modelling power losses and then one has to model with hypotrochoids], and hence operate according to the cross product triangle acting with the properties of complex numbers. The cogs of gears are then the 'roots of unity' around the shaft axis, the order of the roots (of k -cross if the shaft is spinning around the k -hat axis) being the number of cogs around the gear wheel. If the width of the cogs are the same, these are proportional to gear wheel radii, and hence **the number roots of unity** of one cog in relation to **the number of roots of unity** of another gear wheel in which it is in mesh then defines the step-up or step-down in force, similarly as solenoidal transformers that step-up or step-down voltages by meshed lines of magnetic flux. Roots of unity cannot spin concurrently on different axes other than in ways stipulated by Rubiks cubes and similar, which has definite relevance to quantum mechanics.

So note that I am making a distinction between a single-component-basis-cross-product and a general vector-cross-product. Single-component-basis-cross-products (k -cross for example) are complex-number-like whilst general vector-cross-products are quaternion-like (and there is similarly no quaternion field; it is also a completely real R^3 with the $\times$ operator, as I demonstrate in the last section of this document). I refer to the basis cross products as basis **curls** (which **curl** a Rubik's cube slice around 90 degrees) and the cross product to mean the general vector cross product, which isn't possible on a Rubiks cube since an arbitrary vector cross product requires all three rotational degrees of freedom simultaneously. It is the fact that basis vectors are single component unit vectors that is relevant here, here for single component cross products for spin around and dependence upon a single axis, as it is these vectors which have the complex properties of $i = -1$ when appended with $\times$, whilst also spanning and generate the space for R^3 , including the latent axis when working on an R^2 plane [which is still R^3 with the remaining axis latent and ignored but still there].

What is shown by this document, and by the mechanics of Rubiks cubes and derivatives (such as pyraminx and megaminx), is that there is a very extensive mathematics of the basis variables themselves, which tend otherwise to be left in the background to the scalar

functions. So again, the standard basis vectors i -hat, j -hat, and k -hat have the properties of complex numbers ($i = \sqrt{-1}$) once appended with cross $\times$ (which converts them to rotational basis operators).

Basis Curl (Basis Cross-Product) Tables

Table 1: Single basis curl

	$\hat{i}$	$\hat{j}$	$\hat{k}$
$\hat{i} \times$	0	$\hat{k}$	$-\hat{j}$
$\hat{j} \times$	$-\hat{k}$	0	$\hat{i}$
$\hat{k} \times$	$\hat{j}$	$-\hat{i}$	0

These are tables of the three **basis curls** [$\hat{k} \times$, $\hat{j} \times$, and $\hat{i} \times$], and higher order applications: single (first order) **basis curls**, and second and third order basis **curls**. All of these other entries in excess of complex numbers $i = k \times$, i^2 , i^3 , i^4 is how much richer mathematics is going to get with almost all of this applied in physics. Repetitions (powers) of $k \times$, $j \times$, or $i \times$, are similar as **complex numbers** whilst any combination of these are **quaternion like**. These can hence be extended into an arbitrary number of operators by identifying the cycles that permute the basis variables operated on. These then form quaternion like operations, exactly as any permutation of turns on a Rubik's cube forms a group.

It is also proved that the quaternion field is the real R^3 vector volume. A quaternion is a cross product [vector cross-product], that is, a vector of the basis **curls**, i.e., that comprise the standard cross product which is equivalent to three complex numbers, that is, real (in R^3) basis **curls** [which is only possible when there are 3 rotational degrees of freedom which is not the case for a Rubiks cube and, as evidence suggests, not for the very small such as quantum mechanical variables]. The constant can also exist in how a quaternion is defined when it is defined as a cross product operator rather, than a number, and is demonstrated later in this document (hence no such thing as complex 'numbers' or quaternion 'numbers' and field). This is equivalent to adding an identity matrix to the cross product. Note that there should never be a free constant term independent of the 3 basis variables i -hat, j -hat, and k -hat, since these **generate**, **specify**, and **label** a three dimensional volume. A scalar cannot exist within this space independently of one of the three basis variables. Any scalar function must be associated (coupled) with one of the directions if it is to exist in this volume.

Table 2: Double basis curl

	$\hat{i}$	$\hat{j}$	$\hat{k}$
$\hat{i} \times \hat{i} \times$	0	$-\hat{j}$	$-\hat{k}$
$\hat{i} \times \hat{j} \times$	$\hat{j}$	0	0
$\hat{i} \times \hat{k} \times$	$\hat{k}$	0	0
$\hat{j} \times \hat{i} \times$	0	$\hat{i}$	0
$\hat{j} \times \hat{j} \times$	$-\hat{i}$	0	$-\hat{k}$
$\hat{j} \times \hat{k} \times$	0	$\hat{k}$	0
$\hat{k} \times \hat{i} \times$	0	0	$\hat{i}$
$\hat{k} \times \hat{j} \times$	0	0	$\hat{j}$
$\hat{k} \times \hat{k} \times$	$-\hat{i}$	$-\hat{j}$	0

The first highlighted entry corresponds to $i^2 = (\hat{i} \times)^2$ and operates non-redundantly on j -hat and k -hat, the y - z plane i.e., returns their negative since $i^2 = -1$, that is, $(\hat{i} \times)^2(\cdot) = -1(\cdot)$ for j -hat and k -hat (and null for i -hat) $(\hat{i} \times)^2 \hat{j} = \hat{i} \times \hat{i} \times \hat{j} = \hat{i} \times \hat{k} = -\hat{j} = (-1)\hat{j}$ and $(\hat{i} \times)^2 \hat{k} = \hat{i} \times \hat{i} \times \hat{k} = \hat{i} \times (-\hat{j}) = -\hat{k} = (-1)\hat{k}$

The second highlighted entry corresponds to $j^2 = (\hat{j} \times)^2$ and operates non-redundantly on i -hat and k -hat, the x - z plane, i.e., returns their negative since $j^2 = -1$, that is, $(\hat{j} \times)^2(\cdot) = -1(\cdot)$ for i -hat and k -hat (and null for j -hat) $(\hat{j} \times)^2 \hat{i} = \hat{j} \times \hat{j} \times \hat{i} = \hat{j} \times (-\hat{k}) = -\hat{i} = (-1)\hat{i}$ and $(\hat{j} \times)^2 \hat{k} = \hat{j} \times \hat{j} \times \hat{k} = \hat{j} \times \hat{i} = -\hat{k} = (-1)\hat{k}$

This highlighted entry corresponds to $k^2 = (\hat{k} \times)^2$ and operates non-redundantly on i -hat and j -hat, the x - y plane, i.e., returns their negative since $k^2 = -1$, that is, $(\hat{k} \times)^2(\cdot) = -1(\cdot)$ for i -hat and j -hat (and null for k -hat) $(\hat{k} \times)^2 \hat{i} = \hat{k} \times \hat{k} \times \hat{i} = \hat{k} \times \hat{j} = -\hat{i} = (-1)\hat{i}$ and $(\hat{k} \times)^2 \hat{j} = \hat{k} \times \hat{k} \times \hat{j} = \hat{k} \times (-\hat{i}) = -\hat{j} = (-1)\hat{j}$

Table 3: Triple basis curl

	$\hat{i}$	$\hat{j}$	$\hat{k}$
$\hat{i} \times \hat{i} \times \hat{i} \times$	0	$-\hat{k}$	$\hat{j}$
$\hat{i} \times \hat{i} \times \hat{j} \times$	$\hat{k}$	0	0
$\hat{i} \times \hat{i} \times \hat{k} \times$	$-\hat{j}$	0	0
$\hat{i} \times \hat{j} \times \hat{i} \times$	0	0	0
$\hat{i} \times \hat{j} \times \hat{j} \times$	0	0	$\hat{j}$
$\hat{i} \times \hat{j} \times \hat{k} \times$	0	$-\hat{j}$	0
$\hat{i} \times \hat{k} \times \hat{i} \times$	0	0	0
$\hat{i} \times \hat{k} \times \hat{j} \times$	0	0	$\hat{k}$
$\hat{i} \times \hat{k} \times \hat{k} \times$	0	$-\hat{k}$	0

This entry corresponds to i^3 and operates non-redundantly on j -hat and k -hat, i.e., returns their negative and then permutes their axis with an additional cross ($i \times$) around the axis generated by i -hat:

$$(\hat{i} \times)^3 \hat{j} = \hat{i} \times \hat{i} \times \hat{i} \times \hat{j} = \hat{i} \times (-\hat{j}) = (-\hat{i} \times) \hat{j} = -\hat{k} \quad \text{and} \quad (\hat{i} \times)^3 \hat{k} = \hat{i} \times \hat{i} \times \hat{i} \times \hat{k} = \hat{i} \times (-\hat{k}) = (-\hat{i} \times) \hat{k} = \hat{j}$$

This entry corresponds to the quaternion $ijk = -1$ and operates non-redundantly on j -hat only. I'm calling this the '**negative quaternion**': $(\hat{i} \times \hat{j} \times \hat{k} \times) \hat{j} = \hat{i} \times \hat{j} \times \hat{k} \times \hat{j} = \hat{i} \times \hat{j} \times (-\hat{i}) = \hat{i} \times \hat{k} = -\hat{j} = (-1) \hat{j}$

This entry corresponds to the quaternion $ikj = +1$ and operates non-redundantly on k -hat only. I'm calling this the '**positive quaternion**': $(\hat{i} \times \hat{k} \times \hat{j} \times) \hat{k} = \hat{i} \times \hat{k} \times \hat{j} \times \hat{k} = \hat{i} \times \hat{k} \times \hat{i} = \hat{i} \times \hat{j} = \hat{k} = (+1) \hat{k}$

although not actually being a quaternion field since the complex field is definitively equal to a real, rather than complex, vector volume with the binary operator of $\times$. Note that on a Rubiks cube the sequence ikj ($i \times k \times j \times$) does not return the identity. The three move sequence ikj has to be repeated 4 times when all moves are middle layers (through the origin) or 84 times to return the identity, i.e., return the cube to its starting configuration (return all the bits to their **initial positions** and **spin orientations**). If you (for ikj) sway i and j then funnily enough it is 80 times to return to the solved configuration. Try it with a decent Rubiks cube and see for yourself. If you do, think to yourself 'complex numbers' (quartic roots of unity) and 'quaternions' whilst doing the turns and then think about how it is entirely real. This sums up why I wrote this document which I am now extremely pleased with. If there were such a thing as a quaternion field, then it would be equivalent to the fully real R^3 Rubik's cube (Rubiks quaternion); with the very marvellous attribute that you can actually hold it in your hands and physically make the turns and see the orientations change. Brilliant, couldn't recommend it more highly to someone who likes maths and mathematical research.

Table 4: Triple basis curl

	$\hat{i}$	$\hat{j}$	$\hat{k}$
$\hat{j} \times \hat{i} \times \hat{i} \times$	0	0	$-\hat{i}$
$\hat{j} \times \hat{i} \times \hat{j} \times$	0	0	0
$\hat{j} \times \hat{i} \times \hat{k} \times$	$\hat{i}$	0	0
$\hat{j} \times \hat{j} \times \hat{i} \times$	0	$-\hat{k}$	0
$\hat{j} \times \hat{j} \times \hat{j} \times$	$\hat{k}$	0	$-\hat{i}$
$\hat{j} \times \hat{j} \times \hat{k} \times$	0	$\hat{i}$	0
$\hat{j} \times \hat{k} \times \hat{i} \times$	0	0	$-\hat{k}$
$\hat{j} \times \hat{k} \times \hat{j} \times$	0	0	0
$\hat{j} \times \hat{k} \times \hat{k} \times$	$\hat{k}$	0	0

Table 5: Triple basis curl

	$\hat{i}$	$\hat{j}$	$\hat{k}$
$\hat{k} \times \hat{i} \times \hat{i} \times$	0	$\hat{i}$	0
$\hat{k} \times \hat{i} \times \hat{j} \times$	$-\hat{i}$	0	0
$\hat{k} \times \hat{i} \times \hat{k} \times$	0	0	0
$\hat{k} \times \hat{j} \times \hat{i} \times$	0	$\hat{j}$	0
$\hat{k} \times \hat{j} \times \hat{j} \times$	$-\hat{j}$	0	0
$\hat{k} \times \hat{j} \times \hat{k} \times$	0	0	0
$\hat{k} \times \hat{k} \times \hat{i} \times$	0	0	$\hat{j}$
$\hat{k} \times \hat{k} \times \hat{j} \times$	0	0	$-\hat{i}$
$\hat{k} \times \hat{k} \times \hat{k} \times$	$-\hat{j}$	$\hat{i}$	0

Basis Sines and Cosines

For first order discrete curls, there is one and only one basis curl (i.e., complex number), $\hat{k} \times$, $\hat{j} \times$, or $\hat{i} \times$, associated with each trigonometric function, similarly as for exponentials. Hence there are 3 different sines and 3 different cosines (in addition to their scalar forms). These are the $\hat{k} \times$ sines and cosines (**sink** and **cosk**), the $\hat{j} \times$ sines and cosines (**sinj** and **cosj**), and the $\hat{i} \times$ sines and cosines (**sini** and **cosi**). These are similarity invariants (**sink**, **sinj**, and **sini**) and (**cosk**, **cosj**, and **cosi**) because each can be transformed to one of the other three by a change of base whilst also forming closed orthogonal sets because they span the space of R^3 for each of the mutually perpendicular axes. The exponentials and sinusoidals, as operators, end up being independent of $\hat{k} \times$, $\hat{j} \times$, or $\hat{i} \times$ algebraically but not of whatever the spin axis is so are still dependent on the basis curls by way of direction i.e., dependent on what basis vector the trigonometric function (operator) can be associated and couples with. For example, a $\hat{k} \times$ sine function **sink** ends up being independent of $\hat{k} \times$ but must be coupled with $\hat{i}$ and $\hat{j}$ otherwise it degenerates from a harmonic to a linear scalar tied to its basis variable, that is, degenerates to $\hat{k} \theta$ (and $\hat{k}$ for cosine). The k -hat for the k -hat exponential in the k -hat direction is the eigenvalue of the rotation since rotations always have a unity eigenvalue corresponding to the spin axis and this is how it occurs in this context. This axis is the eigenvector of the rotation. This is similarly for (1) the j -hat exponential with j -hat and (2) the i -hat exponential with i -hat given that these are the similarity invariants that form the set of the standard basis for R^3 .

Considering the $\hat{k} \times$ cosine (the k -hat basis cosine). Expressing the $\hat{k} \times$ **dependent cosine** in terms of the k -hat exponentials, the k -hat exponential $e^{\theta \hat{k} \times}$ and its conjugate $e^{-\theta \hat{k} \times}$ to make the definition, initially in terms of hyperbolics so that I can keep track of the $\hat{k} \times$, for what I am calling the **k -hat basis cosine (cosk)**

$$\text{cosk} \theta = \cosh(\theta \hat{k} \times (\cdot)) = \cosh(\theta \hat{k} \times) = \frac{1}{2} (e^{\theta \hat{k} \times} + e^{-\theta \hat{k} \times})$$

and, as James Gregory did almost 400 years ago for the scalar sine and cosine, expanding in a power series

$$\begin{aligned} \text{cosk} \theta &= \frac{1}{2} (e^{\theta \hat{k} \times} + e^{-\theta \hat{k} \times}) = \frac{1}{2} (e^{\theta \hat{k} \times}) - \frac{1}{2} (e^{-\theta \hat{k} \times}) \\ &= \frac{1}{2} (1 + \theta(\hat{k} \times) + \frac{1}{2!} \theta^2 (\hat{k} \times)^2 + \frac{1}{3!} \theta^3 (\hat{k} \times)^3 + \frac{1}{4!} \theta^4 (\hat{k} \times)^4 + \dots) \\ &\quad + \frac{1}{2} (1 - \theta(\hat{k} \times) + \frac{1}{2!} \theta^2 (\hat{k} \times)^2 - \frac{1}{3!} \theta^3 (\hat{k} \times)^3 + \frac{1}{4!} \theta^4 (\hat{k} \times)^4 + \dots) \\ &= 1 + \frac{1}{2!} \theta^2 (\hat{k} \times)^2 + \frac{1}{4!} \theta^4 (\hat{k} \times)^4 + \dots \end{aligned}$$

and when it is understood that the cosine is associated with $\hat{i}$ or $\hat{j}$, then $(\hat{k} \times)^2 (\cdot) = -1(\cdot)$ and $(\hat{k} \times)^4 (\cdot) = 1(\cdot)$ so that

$$\text{cosk} \theta = \cosh(\theta \hat{k} \times (\cdot)) = \cosh(\theta \hat{k} \times) = 1 - \frac{1}{2!} \theta^2 + \frac{1}{4!} \theta^4 - \dots$$

which is the usual power series expansion of cosine and is algebraically independent of $\hat{k} \times$. But it is understood that this cosine (the $\hat{k} \times$ basis cosine) is always associated with a direction, either $\hat{i}$ or $\hat{j}$, so that any power of $\hat{k} \times$ does not have meaning outside of a coupling with a direction. For instance, for the $\hat{k} \times$ cosine in the $\hat{i}$ direction

$(\hat{k} \times)^2 \hat{i} = \hat{k} \times (\hat{k} \times \hat{i}) = \hat{k} \times (\hat{j}) = \hat{k} \times \hat{j} = -\hat{i}$ and $(\hat{k} \times)^4 \hat{i} = \hat{i}$ so that

$$\begin{aligned}
 \mathbf{cosk} \theta \hat{i} &= \left(1 + \frac{1}{2!} \theta^2 (\hat{k} \times)^2 + \frac{1}{4!} \theta^4 (\hat{k} \times)^4 + \dots \right) \hat{i} \\
 &= \hat{i} + \frac{1}{2!} \theta^2 (\hat{k} \times)^2 \hat{i} + \frac{1}{4!} \theta^4 (\hat{k} \times)^4 \hat{i} + \dots \\
 &= \hat{i} - \frac{1}{2!} \theta^2 \hat{i} + \frac{1}{4!} \theta^4 \hat{i} - \dots \\
 &= \left(1 - \frac{1}{2!} \theta^2 + \frac{1}{4!} \theta^4 - \dots \right) \hat{i} \\
 &= \hat{i} \left(1 - \frac{1}{2!} \theta^2 + \frac{1}{4!} \theta^4 - \dots \right) \\
 &= \hat{i} \cos \theta
 \end{aligned}$$

since $\left(1 - \frac{1}{2!} \theta^2 + \frac{1}{4!} \theta^4 - \dots \right) = \cos \theta$ is scalar and hence commutes with i -hat

so it is exactly the same as previously, a scalar cosine, but coupled with the $\hat{i}$ direction. Similarly for the $\hat{k} \times$ basis cosine in the $\hat{j}$ direction

$$\begin{aligned}
 \mathbf{cosk} \theta \hat{j} &= \left(1 + \frac{1}{2!} \theta^2 (\hat{k} \times)^2 + \frac{1}{4!} \theta^4 (\hat{k} \times)^4 + \dots \right) \hat{j} \\
 &= \hat{j} + \frac{1}{2!} \theta^2 (\hat{k} \times)^2 \hat{j} + \frac{1}{4!} \theta^4 (\hat{k} \times)^4 \hat{j} + \dots \\
 &= \hat{j} - \frac{1}{2!} \theta^2 \hat{j} + \frac{1}{4!} \theta^4 \hat{j} - \dots \\
 &= \left(1 - \frac{1}{2!} \theta^2 + \frac{1}{4!} \theta^4 - \dots \right) \hat{j} \\
 &= \hat{j} \left(1 - \frac{1}{2!} \theta^2 + \frac{1}{4!} \theta^4 - \dots \right) \\
 &= \hat{j} \cos \theta
 \end{aligned}$$

since $\left(1 - \frac{1}{2!} \theta^2 + \frac{1}{4!} \theta^4 - \dots \right) = \cos \theta$ is scalar and hence commutes with j -hat

since $(\hat{k} \times)^2 \hat{j} = \hat{k} \times (\hat{k} \times \hat{j}) = \hat{k} \times (-\hat{i}) = -\hat{k} \times \hat{i} = \hat{j}$ and $(\hat{k} \times)^4 \hat{j} = \hat{j}$, which is the scalar cosine in the j -hat direction.

That was interesting, but now for the really interesting part. When the $\hat{k} \times$ cosine is associated with the $\hat{k}$ direction, all terms in $\hat{k} \times$ zero and **the harmonic properties degenerate**. What is left is essentially the small angle approximation, here it is its direction on the $\hat{k}$ axis. This corresponds to what linear algebra consider the eigenvalue associated with the unitary eigenvector

$$\begin{aligned}
 \mathbf{cosk} \theta \hat{k} &= \left(1 + \frac{1}{2!} \theta^2 (\hat{k} \times)^2 + \frac{1}{4!} \theta^4 (\hat{k} \times)^4 + \dots \right) \hat{k} \\
 &= \hat{k} + \frac{1}{2!} \theta^2 (\hat{k} \times)^2 \hat{k} + \frac{1}{4!} \theta^4 (\hat{k} \times)^4 \hat{k} + \dots \\
 &= \hat{k} - \frac{1}{2!} \theta^2 (-0\hat{k}) + \frac{1}{4!} \theta^4 (0\hat{k}) - \dots \\
 &= \left(1 - \frac{1}{2!} \theta^2 (0) + \frac{1}{4!} \theta^4 (0) - \dots \right) \hat{k} \\
 &= (1 - 0 + 0 - 0 + 0 - \dots) \hat{k} \\
 &= \hat{k} (1 - 0 + 0 - 0 + 0 - \dots) \\
 &= \hat{k}
 \end{aligned}$$

These results (for **cosk**) and the obvious inferred similarity invariants for the j -hat and i -hat axes (**cosj** and **cosi**) that comprise the standard basis, proves definitively that there are three different sines and three different cosines [and the scalars], one each associated with their respective basis curl and rotation axis. These are the k -hat cosine **cosk**, the j -hat cosine **cosj**, and the i -hat cosine **cosi** and

these are operators similarly as proved above for **cosk**. Similarly for sine as shown next. In summary, for the k -hat cosine (the $\hat{k} \times$ cosine), those which are associated with $\exp(\theta \hat{k} \times (\cdot)) = \exp(\theta k \times)$, coupled with each direction are

$$(\mathbf{cosk}\theta) \hat{k} = \cosh(\theta \hat{k} \times (\cdot)) \hat{k} = \cosh(\theta \hat{k} \times) \hat{k} = (1 - 0 + 0 - \dots) \hat{k} = \hat{k} (1 - 0 + 0 - \dots) = \hat{k} \quad (\text{"cos-kay theta kay-hat"})$$

$$(\mathbf{cosk}\theta) \hat{j} = \cosh(\theta \hat{k} \times (\cdot)) \hat{j} = \cosh(\theta \hat{k} \times) \hat{j} = (1 - \frac{1}{2!} \theta^2 + \frac{1}{4!} \theta^4 - \dots) \hat{j} = \hat{j} (1 - \frac{1}{2!} \theta^2 + \frac{1}{4!} \theta^4 - \dots) = \hat{j} \cos \theta \quad (\text{"cos-kay theta jay-hat"})$$

$$(\mathbf{cosk}\theta) \hat{i} = \cosh(\theta \hat{k} \times (\cdot)) \hat{i} = \cosh(\theta \hat{k} \times) \hat{i} = (1 - \frac{1}{2!} \theta^2 + \frac{1}{4!} \theta^4 - \dots) \hat{i} = \hat{i} (1 - \frac{1}{2!} \theta^2 + \frac{1}{4!} \theta^4 - \dots) = \hat{i} \cos \theta \quad (\text{"cos-kay theta eye-hat"})$$

noting where the commutations occur. Noted above is also how you say them (the left hand side before operating on the basis hat) when doing maths with these operators. This is included since it is new and unfamiliar and contrasts with the familiar $\cosh \theta$ (**kosh theta**). These equations are, in words

cos-kay theta kay-hat (before basis-hat operation) = **kay-hat** (after basis-hat operation)

cos-kay theta jay-hat (before basis-hat operation) = **jay-hat cos theta** (after basis-hat operation)

cos-kay theta eye-hat (before basis-hat operation) = **eye-hat cos theta** (after basis-hat operation)

These were in terms of hyperbolics to keep track of the $\hat{k} \times$ which are ordinarily only explicit in exponential form as $i = \sqrt{-1}$ in ordinary mathematics. Now for the sines.

Considering the $\hat{k} \times$ sine (the k -hat basis sine). The $\hat{k} \times$ sine, that comprised is of the k -hat exponential $e^{\theta \hat{k} \times}$ and its conjugate $e^{-\theta \hat{k} \times}$. When coupled with each direction are similar as the cosines. Here I am initially, similarly as cosine, defining in terms of hyperbolics for what I am calling the **k -hat basis sine** (and denoting **sink**)

$$\mathbf{sink}\theta = (\hat{k} \times (\cdot))^{-1} \sinh(\theta \hat{k} \times (\cdot)) = (\hat{k} \times)^{-1} \sinh(\theta k \times) = (\hat{k} \times)^{-1} \frac{1}{2} (e^{\theta \hat{k} \times} - e^{-\theta \hat{k} \times}) = (\hat{k} \times)^{-1} \left(\frac{1}{2} (e^{\theta \hat{k} \times}) - \frac{1}{2} (e^{-\theta \hat{k} \times}) \right)$$

Expanding the k -hat basis sine (the $\hat{k} \times$ sine) in a power series

$$\begin{aligned} \mathbf{sink}\theta &= (\hat{k} \times)^{-1} \left(\frac{1}{2} (e^{\theta \hat{k} \times}) - \frac{1}{2} (e^{-\theta \hat{k} \times}) \right) \\ &= (\hat{k} \times)^{-1} \left(\frac{1}{2} (1 + \theta(\hat{k} \times) + \frac{1}{2!} \theta^2 (\hat{k} \times)^2 + \frac{1}{3!} \theta^3 (\hat{k} \times)^3 + \frac{1}{4!} \theta^4 (\hat{k} \times)^4 + \dots) \right. \\ &\quad \left. - \frac{1}{2} (1 - \theta(\hat{k} \times) + \frac{1}{2!} \theta^2 (\hat{k} \times)^2 - \frac{1}{3!} \theta^3 (\hat{k} \times)^3 + \frac{1}{4!} \theta^4 (\hat{k} \times)^4 - \dots) \right) \\ &= (\hat{k} \times)^{-1} \left(\theta(\hat{k} \times) + \frac{1}{3!} \theta^3 (\hat{k} \times)^3 + \frac{1}{5!} \theta^5 (\hat{k} \times)^5 + \dots \right) \\ &= \theta + \frac{1}{3!} \theta^3 (\hat{k} \times)^2 + \frac{1}{5!} \theta^5 (\hat{k} \times)^4 + \dots \end{aligned}$$

as for the cosine, when it is understood that the sine is associated with $\hat{i}$ or $\hat{j}$, then $(\hat{k} \times)^2 = -1$ and $(\hat{k} \times)^4 = 1$ so that

$$\mathbf{sink}\theta = (\hat{k} \times (\cdot))^{-1} \sinh(\theta \hat{k} \times (\cdot)) = (\hat{k} \times)^{-1} \sinh(\theta k \times) = \theta - \frac{1}{3!} \theta^3 + \frac{1}{5!} \theta^5 - \dots$$

which is the usual power series expansion of sine and is algebraically independent of $\hat{k} \times$ similarly as the cosine.

This sine, the k -hat or k -cross sine (the $\hat{k} \times$ sine), is similarly as the cosine, always associated with a direction, either $\hat{i}$ or $\hat{j}$, so that any power $\hat{k} \times$ does not have meaning outside of a coupling with a direction. For the $\hat{k} \times$ sine in the $\hat{i}$ direction, as for the cosine

$$(\hat{k} \times)^2 \hat{i} = \hat{k} \times (\hat{k} \times \hat{i}) = \hat{k} \times (\hat{j}) = \hat{k} \times \hat{j} = -\hat{i} \quad \text{and} \quad (\hat{k} \times)^4 \hat{i} = \hat{i}$$

so that

$$\begin{aligned} (\text{sink } \theta) \hat{i} &= (\hat{k} \times)^{-1} \left(\theta (\hat{k} \times) + \frac{1}{3!} \theta^3 (\hat{k} \times)^3 + \frac{1}{5!} \theta^5 (\hat{k} \times)^5 + \dots \right) \hat{i} \\ &= \left(\theta + \frac{1}{3!} \theta^3 (\hat{k} \times)^2 + \frac{1}{5!} \theta^5 (\hat{k} \times)^4 + \dots \right) \hat{i} \\ &= \theta \hat{i} + \frac{1}{3!} \theta^3 (\hat{k} \times)^2 \hat{i} + \frac{1}{5!} \theta^5 (\hat{k} \times)^4 \hat{i} + \dots \\ &= \theta \hat{i} - \frac{1}{3!} \theta^3 \hat{i} + \frac{1}{5!} \theta^5 \hat{i} - \dots && \text{since } \left(\theta - \frac{1}{3!} \theta^3 + \frac{1}{5!} \theta^5 - \dots \right) = \sin \theta \text{ is scalar and hence commutes} \\ &= \left(\theta - \frac{1}{3!} \theta^3 + \frac{1}{5!} \theta^5 - \dots \right) \hat{i} \\ &= \hat{i} \left(\theta - \frac{1}{3!} \theta^3 + \frac{1}{5!} \theta^5 - \dots \right) \\ &= \hat{i} \sin \theta \end{aligned}$$

so it is exactly the same as previously above, and as for cosine, but coupled with the $\hat{i}$ direction. Similarly for the $\hat{k} \times$ sine in the $\hat{j}$ direction,

$$\begin{aligned} (\text{sink } \theta) \hat{j} &= (\hat{k} \times)^{-1} \left(\theta (\hat{k} \times) + \frac{1}{3!} \theta^3 (\hat{k} \times)^3 + \frac{1}{5!} \theta^5 (\hat{k} \times)^5 + \dots \right) \hat{j} \\ &= \left(\theta + \frac{1}{3!} \theta^3 (\hat{k} \times)^2 + \frac{1}{5!} \theta^5 (\hat{k} \times)^4 + \dots \right) \hat{j} \\ &= \theta \hat{j} + \frac{1}{3!} \theta^3 (\hat{k} \times)^2 \hat{j} + \frac{1}{5!} \theta^5 (\hat{k} \times)^4 \hat{j} + \dots \\ &= \theta \hat{j} - \frac{1}{3!} \theta^3 \hat{j} + \frac{1}{5!} \theta^5 \hat{j} - \dots \\ &= \left(\theta - \frac{1}{3!} \theta^3 + \frac{1}{5!} \theta^5 - \dots \right) \hat{j} \\ &= \hat{j} \left(\theta - \frac{1}{3!} \theta^3 + \frac{1}{5!} \theta^5 - \dots \right) \\ &= \hat{j} \sin \theta \end{aligned}$$

For the $\hat{k} \times$ sine in the $\hat{k}$ direction, its own direction, all of the $\hat{k} \times$ cancel whenever it couples with its own basis vector $\hat{k}$ and the harmonic degenerates upon coupling with and operating on the k -hat basis variable into the small angle approximation for sine in the $\hat{k}$ direction

$$\begin{aligned}
\text{sink } \theta \hat{k} &= (\hat{k} \times)^{-1} \left(\theta (\hat{k} \times) + \frac{1}{3!} \theta^2 (\hat{k} \times)^3 + \frac{1}{5!} \theta^5 (\hat{k} \times)^5 + \dots \right) \hat{k} \\
&= \left(\theta + \frac{1}{3!} \theta^3 (\hat{k} \times)^2 + \frac{1}{5!} \theta^5 (\hat{k} \times)^4 + \dots \right) \hat{k} \\
&= \theta \hat{k} + \frac{1}{3!} \theta^3 (\hat{k} \times)^2 \hat{k} + \frac{1}{5!} \theta^5 (\hat{k} \times)^4 \hat{k} + \dots \\
&= \theta \hat{k} + \frac{1}{3!} \theta^3 (-0 \hat{k}) + \frac{1}{5!} \theta^5 (0 \hat{k}) - \dots \\
&= \left(\theta \hat{k} - \frac{1}{3!} \theta^3 (0) + \frac{1}{5!} \theta^5 (0) - \dots \right) \hat{k} \\
&= (\theta - 0 + 0 - \dots) \hat{k} \\
&= \hat{k} (\theta - 0 + 0 - \dots) \\
&= \theta \hat{k}
\end{aligned}$$

This is probably one important reason why the small angle approximation for sine is useful in physics. This essentially boils down to what we know already... it linearizes the harmonic (which is why it is useful, because it simplifies the mathematics). So, in summary, for the k -hat basis sine (the $\hat{k} \times$ sine), that associated with $\exp(\theta \hat{k} \times (\cdot)) = \exp(\theta \hat{k} \times)$, coupled with each direction

$$(\text{sink } \theta) \hat{k} = (\hat{k} \times)^{-1} \sinh(\theta \hat{k} \times) \hat{k} = (\theta - 0 + 0 - \dots) \hat{k} = \hat{k} (\theta - 0 + 0 - \dots) = \hat{k} \theta \quad (\text{sine-kay theta kay-hat})$$

$$(\text{sink } \theta) \hat{j} = (\hat{k} \times)^{-1} \sinh(\theta \hat{k} \times) \hat{j} = \left(\theta - \frac{1}{3!} \theta^3 + \frac{1}{5!} \theta^5 - \dots \right) \hat{j} = \hat{j} \left(\theta - \frac{1}{3!} \theta^3 + \frac{1}{5!} \theta^5 - \dots \right) = \hat{j} \sin \theta \quad (\text{sine-kay theta jay-hat})$$

$$(\text{sink } \theta) \hat{i} = (\hat{k} \times)^{-1} \sinh(\theta \hat{k} \times) \hat{i} = \left(\theta - \frac{1}{3!} \theta^3 + \frac{1}{5!} \theta^5 - \dots \right) \hat{i} = \hat{i} \left(\theta - \frac{1}{3!} \theta^3 + \frac{1}{5!} \theta^5 - \dots \right) = \hat{i} \sin \theta \quad (\text{sine-kay theta eye-hat})$$

sine-kay theta kay-hat (before basis-hat operation) = **kay-hat theta** (after basis-hat operation)

sine-kay theta jay-hat (before basis-hat operation) = **jay-hat sine theta** (after basis-hat operation)

sine-kay theta eye-hat (before basis-hat operation) = **eye-hat sine theta** (after basis-hat operation)

In summary for the k -hat sines and cosines: in light of the analysis of the k -hat exponential, the commutation properties, its Euler representation (considered next) from power series expansions, and its coupling to the basis vectors, it came to my attention (I become aware of/discovered in searching for the sphere) that there is actually four types of sine and four types of cosine: the three types of basis sine and cosine that are tied with their basis-hat, and the sine and the cosine that have been operated on by (coupled with) a direction basis vector and are hence basis-hat independent and consequently ordinary scalar sine and ordinary scalar cosine. Those that are tied with their basis-hat vector are **operators** and hence **do not commute** (other than with operators of the same basis-hat). Those that have been operated on, and are hence basis-hat independent, are **scalars** and hence **do commute**. These are the standard sines and cosines from algebra and trigonometry, and for which I refer to as the **scalar sines and cosines**.

I have denoted (with backtracking) those which still depend on the basis vector associated with the complex exponential as **sink**, **sinj**, **sini**, **cosk**, **cosj**, and **cosi**. I have denoted these functions in this way for several reasons but mostly to keep track of the basis dependence while doing maths with them (that is, keeping track of which ones are associated with k -hat, which ones with j -hat, and which ones with i -hat). Once operated on a basis vector, they become **basis-hat invariant** and revert to scalars (tied to a direction) and hence the ordinary form of sine and cosine, denoted as ordinary sine and cosine, **sin** and **cos**. Again, this means that the basis vector independent (scalar) sines and cosines (**sin** and **cos**) commute but the others, the basis dependent operators (**sink**, **sinj**, **sini**, **cosk**, **cosj**, and **cosi**), do not. They do not commute with each other (unless as a function of the same axis) or with the basis vectors since these are what they operate (right to left) on. Operator goes on the left, operand goes on the right.

Hence, there are four sines and four cosines. Three of the **sines** are operators (which hence are not commutative) and one, the usual one, is a scalar (and hence arbitrarily commutes). Three of the **cosines** are operators and one, the usual one similarly as sine, is a scalar. Each of the operators form a couple with a basis vector, two such couples covert the basis-hat dependent sine or cosine to the scalar and the remaining couple, that with which the operator is based, degenerates and reverts to either the basis variable itself (cosine) or the angle/argument coupled with the basis variable (sine). The complete enumeration of these (with the **operation commutations**) are hence tabulated:

Figure 6 Summary: k -hat, j -hat, and i -hat sines and cosines

Scalar function	k -hat operators	j -hat operators	i -hat operators
$\sin \theta$	$(\mathbf{sink} \theta) \hat{k} = \hat{k} \theta$	$(\mathbf{sinj} \theta) \hat{k} = \hat{k} \sin \theta$	$(\mathbf{sini} \theta) \hat{k} = \hat{k} \sin \theta$
	$(\mathbf{sink} \theta) \hat{j} = \hat{j} \sin \theta$	$(\mathbf{sinj} \theta) \hat{j} = \hat{j} \theta$	$(\mathbf{sini} \theta) \hat{j} = \hat{j} \sin \theta$
	$(\mathbf{sink} \theta) \hat{i} = \hat{i} \sin \theta$	$(\mathbf{sinj} \theta) \hat{i} = \hat{i} \sin \theta$	$(\mathbf{sini} \theta) \hat{i} = \hat{i} \theta$
$\cos \theta$	$(\mathbf{cosk} \theta) \hat{k} = \hat{k}$	$(\mathbf{cosj} \theta) \hat{k} = \hat{k} \cos \theta$	$(\mathbf{cosi} \theta) \hat{k} = \hat{k} \cos \theta$
	$(\mathbf{cosk} \theta) \hat{j} = \hat{j} \cos \theta$	$(\mathbf{cosj} \theta) \hat{j} = \hat{j}$	$(\mathbf{cosi} \theta) \hat{j} = \hat{j} \cos \theta$
	$(\mathbf{cosk} \theta) \hat{i} = \hat{i} \cos \theta$	$(\mathbf{cosj} \theta) \hat{i} = \hat{i} \cos \theta$	$(\mathbf{cosi} \theta) \hat{i} = \hat{i}$

Basis Exponential and Vector Basis Euler's Theorem

Here is a vector basis form of Euler's theorem which was (I) developed specifically for the research that I describe in the next topic regarding the Riemann sphere. The vector basis form of Euler's theorem pertains to what I have developed for the k -hat basis exponential **expk**, which expresses the k -hat exponential **expk** in terms of the sinusoidal operators **cosk** and **sink**. Hence, the k -hat exponential **expk** is similarly an operator that operates on the basis vectors, similarly as **cosk** and **sink**, and there are three such isometric exponential operators for R^3 (the k -hat exponential **expk**, the j -hat exponential **expj**, and the i -hat exponential **expi**). Ultimately, this is the full form of the single component cross product basis exponential as a vector operator, expressible as and equivalent to the rotation matrices (R_z for **expk**, R_y for **expj**, and R_x for **expi**) and complex exponential (with $i = k \times$, $j \times$, and $i \times$ for **expk**, **expj**, and **expi**), and becomes a circular vector [parametric circle] once it has been applied to (has operated on) a coordinate vector; noting that the values of the coordinate vector determine the radius and phase of the circle. The basis exponential takes the form of one of the three aforementioned isometric types; the k -hat exponential **expk**, the j -hat exponential **expj**, and the i -hat exponential **expi**; here will discuss the k -hat basis exponential **expk**, whilst that of **expj** and **expi** have similar mathematics given that **expj** and **expi** are isometry invariants of **expk**.

The power series expansion of the k -hat exponential, with an unspecified direction relying solely on the properties of $\sqrt{-1}$, is

$$\begin{aligned}
 e^{\theta \hat{k} \times} &= \exp(\theta \hat{k} \times (\cdot)) \\
 &= 1 + \theta (\hat{k} \times) + \frac{1}{2!} \theta^2 (\hat{k} \times)^2 + \frac{1}{3!} \theta^3 (\hat{k} \times)^3 + \frac{1}{4!} \theta^4 (\hat{k} \times)^4 + \dots \\
 &= 1 + \theta (\hat{k} \times) - \frac{1}{2!} \theta^2 - \frac{1}{3!} \theta^3 (\hat{k} \times) + \frac{1}{4!} \theta^4 + \dots
 \end{aligned}$$

which, when the odd and even components of the expansion are separated, is

$$\begin{aligned}
 e^{\theta \hat{k} \times} &= \exp(\theta \hat{k} \times (\cdot)) \\
 &= \left(1 - \frac{1}{2!} \theta^2 + \frac{1}{4!} \theta^4 - \dots\right) + \left(\theta(\hat{k} \times) - \frac{1}{3!} \theta^3(\hat{k} \times) + \frac{1}{5!} \theta^5(\hat{k} \times) - \dots\right) \\
 &= \left(1 - \frac{1}{2!} \theta^2 + \frac{1}{4!} \theta^4 - \dots\right) + \left(\theta - \frac{1}{3!} \theta^3 + \frac{1}{5!} \theta^5 - \dots\right)(\hat{k} \times) \\
 &= \cos \theta + (\sin \theta) \hat{k} \times (\cdot)
 \end{aligned}$$

noting that since the cosine and sine have not yet been applied to a basis direction, these are still operators (latently) that depend on even powers of $\hat{k} \times$ (and odd powers for sine with the explicit lagging $\hat{k} \times$ which keeps that term non-singular), so I have decided to denote this as (which is the beautiful part and why I first realised the need to define the operators **cosk** and **sink**)

$$\begin{aligned}
 e^{\theta \hat{k} \times} &= \exp(\theta \hat{k} \times (\cdot)) \\
 &= 1 + \theta(\hat{k} \times) + \frac{1}{2!} \theta^2 (\hat{k} \times)^2 + \frac{1}{3!} \theta^3 (\hat{k} \times)^3 + \frac{1}{4!} \theta^4 (\hat{k} \times)^4 + \dots \\
 &= \left(1 + \frac{1}{2!} \theta^2 (\hat{k} \times)^2 + \frac{1}{4!} \theta^4 (\hat{k} \times)^4 + \dots\right) + \left(\theta(\hat{k} \times) + \frac{1}{3!} \theta^3 (\hat{k} \times)^3 + \frac{1}{5!} \theta^5 (\hat{k} \times)^5 + \dots\right) \\
 &= \mathbf{cosk} \theta + (\mathbf{sink} \theta) \hat{k} \times (\cdot)
 \end{aligned}$$

where $\mathbf{cosk} \theta = 1 + \frac{1}{2!} \theta^2 (\hat{k} \times)^2 + \frac{1}{4!} \theta^4 (\hat{k} \times)^4 + \dots$ and $(\mathbf{sink} \theta) \hat{k} \times = \theta(\hat{k} \times) + \frac{1}{3!} \theta^3 (\hat{k} \times)^3 + \frac{1}{5!} \theta^5 (\hat{k} \times)^5 + \dots$ which then applies as an operator to any of the three standard basis vectors, and is in the form of Euler's equation here for a standard vector basis where the bold font of **cosk** and **sink** indicates dependence on $\times$ and the appended "kay" of "cosk-kay" and "sin-kay" indicates $\hat{k}$ dependence, and hence that these functions are operators specifically as functions of $\hat{k} \times$ that operate on the standard vector basis. Hence, the new mathematical term of *vector basis Euler formula*.

When specified in the x direction with i -hat

$$\begin{aligned}
 e^{\theta \hat{k} \times} \hat{i} &= \exp(\theta \hat{k} \times (\cdot)) \hat{i} \\
 &= \left(1 + \theta(\hat{k} \times) + \frac{1}{2!} \theta^2 (\hat{k} \times)^2 + \frac{1}{3!} \theta^3 (\hat{k} \times)^3 + \frac{1}{4!} \theta^4 (\hat{k} \times)^4 + \dots\right) \hat{i} \\
 &= \hat{i} + \theta(\hat{k} \times) \hat{i} + \frac{1}{2!} \theta^2 (\hat{k} \times)^2 \hat{i} + \frac{1}{3!} \theta^3 (\hat{k} \times)^3 \hat{i} + \frac{1}{4!} \theta^4 (\hat{k} \times)^4 \hat{i} + \dots \\
 &= \hat{i} + \theta(\hat{k} \times \hat{i}) - \frac{1}{2!} \theta^2 \hat{i} - \frac{1}{3!} \theta^3 (\hat{k} \times \hat{i}) + \frac{1}{4!} \theta^4 \hat{i} + \dots \\
 &= \hat{i} + \theta \hat{j} - \frac{1}{2!} \theta^2 \hat{i} + \frac{1}{3!} \theta^3 \hat{j} + \frac{1}{4!} \theta^4 \hat{j} + \dots \\
 &= \hat{i} \left(1 - \frac{1}{2!} \theta^2 + \frac{1}{4!} \theta^4 + \dots\right) + \hat{j} \left(\theta - \frac{1}{3!} \theta^3 + \frac{1}{5!} \theta^5 - \dots\right) \\
 &= \hat{i} \cos \theta + \hat{j} \sin \theta
 \end{aligned}$$

that is,

$$e^{\theta \hat{k} \times} \hat{i} = \exp(\theta \hat{k} \times (\cdot)) \hat{i} = (\mathbf{cosk} \theta + (\mathbf{sink} \theta) \hat{k} \times) \hat{i} = (\mathbf{cosk} \theta) \hat{i} + (\mathbf{sink} \theta) \hat{k} \times \hat{i} = (\mathbf{cosk} \theta) \hat{i} + (\mathbf{sink} \theta) \hat{j} = \hat{i} (\cos \theta) + \hat{j} (\sin \theta)$$

and when specified in the y direction with j -hat

$$\begin{aligned}
e^{\theta \hat{k} \times} \hat{j} &= \exp(\theta \hat{k} \times (\cdot)) \hat{j} \\
&= \left(1 + \theta(\hat{k} \times) + \frac{1}{2!} \theta^2 (\hat{k} \times)^2 + \frac{1}{3!} \theta^3 (\hat{k} \times)^3 + \frac{1}{4!} \theta^4 (\hat{k} \times)^4 + \dots\right) \hat{j} \\
&= \hat{j} + \theta(\hat{k} \times) \hat{j} + \frac{1}{2!} \theta^2 (\hat{k} \times)^2 \hat{j} + \frac{1}{3!} \theta^3 (\hat{k} \times)^3 \hat{j} + \frac{1}{4!} \theta^4 (\hat{k} \times)^4 \hat{j} + \dots \\
&= \hat{j} + \theta(\hat{k} \times \hat{j}) - \frac{1}{2!} \theta^2 \hat{j} - \frac{1}{3!} \theta^3 (\hat{k} \times \hat{j}) + \frac{1}{4!} \theta^4 \hat{j} + \dots \\
&= \hat{j} - \theta \hat{i} - \frac{1}{2!} \theta^2 \hat{j} + \frac{1}{3!} \theta^3 \hat{i} + \frac{1}{4!} \theta^4 \hat{j} + \dots \\
&= \hat{j} \left(1 - \frac{1}{2!} \theta^2 + \frac{1}{4!} \theta^4 + \dots\right) - \hat{i} \left(\theta - \frac{1}{3!} \theta^3 + \frac{1}{5!} \theta^5 - \dots\right) \\
&= \hat{j} \cos \theta - \hat{i} \sin \theta
\end{aligned}$$

similarly, when $e^{\theta \hat{k} \times} = \exp(\theta \hat{k} \times (\cdot))$ is specified in the y -direction with $\hat{j}$ (the k -hat exponential in the j -hat direction), then

$$e^{\theta \hat{k} \times} \hat{j} = \exp(\theta \hat{k} \times (\cdot)) \hat{j} = (\mathbf{cosk} \theta + (\mathbf{sink} \theta) \hat{k} \times) \hat{j} = (\mathbf{cosk} \theta) \hat{j} + (\mathbf{sink} \theta) \hat{k} \times \hat{j} = (\mathbf{cosk} \theta) \hat{j} - (\mathbf{sink} \theta) \hat{i} = -\hat{i} (\sin \theta) + \hat{j} (\cos \theta)$$

which is the 90° phase shifted circle of the k -hat exponential in the i -hat direction.

Here is **the interesting point as for sines and cosines** and why it differs from the sole properties of $\sqrt{-1}$, that regarding the k -hat exponential in the k -hat direction, that is, the k -hat exponential pointing in its own direction. The component harmonics degenerate so that it reverts to its own basis direction, that is, the eigenvector of the circulation: $e^{\theta \hat{k} \times} = \exp(\theta \hat{k} \times (\cdot))$ is specified in the z -direction with $\hat{k}$

$$\begin{aligned}
e^{\theta \hat{k} \times} \hat{k} &= \exp(\theta \hat{k} \times (\cdot)) \hat{k} \\
&= \left(1 + \theta(\hat{k} \times) + \frac{1}{2!} \theta^2 (\hat{k} \times)^2 + \frac{1}{3!} \theta^3 (\hat{k} \times)^3 + \frac{1}{4!} \theta^4 (\hat{k} \times)^4 + \dots\right) \hat{k} \\
&= \hat{k} + \theta(\hat{k} \times) \hat{k} + \frac{1}{2!} \theta^2 (\hat{k} \times)^2 \hat{k} + \frac{1}{3!} \theta^3 (\hat{k} \times)^3 \hat{k} + \frac{1}{4!} \theta^4 (\hat{k} \times)^4 \hat{k} + \dots \\
&= \hat{k} + \theta(\hat{k} \times \hat{k}) + \frac{1}{2!} \theta^2 (\hat{k} \times \hat{k}) + \frac{1}{3!} \theta^3 (\hat{k} \times \hat{k}) + \frac{1}{4!} \theta^4 (\hat{k} \times \hat{k}) + \dots \\
&= \hat{k} (1 + \theta(0) + \frac{1}{2!} \theta^2 (0) + \frac{1}{3!} \theta^3 (0) + \frac{1}{4!} \theta^4 (0) + \dots) \\
&= \hat{k} (1 + 0 + 0 + 0 + 0 + \dots) \\
&= \hat{k} 1 \quad (\text{unitary rotation eigenvalue } +1 \text{ and unitary rotation eigenvector } \hat{k})
\end{aligned}$$

that is,

$$e^{\theta \hat{k} \times} \hat{k} = \exp(\theta \hat{k} \times (\cdot)) \hat{k} = (\mathbf{cosk} \theta) \hat{k} + (\mathbf{sink} \theta) \hat{k} \times \hat{k} = (\mathbf{cosk} \theta) \hat{k} + (\mathbf{sink} \theta) (0) = \hat{k}$$

the last bit of working arising from the k -hat dependent cosine $(\mathbf{cosk} \theta) \hat{k} = \hat{k}$ which is the unity eigenvector (k -hat) of the **expk** rotation with the unit eigenvalue (+1 coefficient of k -hat).

Note again that by the definitions of **cosk**, **cosj**, and **cosi** as cross product operators as functions of the standard single component basis vectors appended with $\times$, $\hat{k} \times$, $\hat{j} \times$, and $\hat{i} \times$; whilst $(\mathbf{cosk} \theta) \hat{k} = \hat{k}$, those of **cosj** and **cosi** operating left-to-right on k -hat are $(\mathbf{cosj} \theta) \hat{k} = \hat{k} \cos \theta$ and $(\mathbf{cosi} \theta) \hat{k} = \hat{k} \cos \theta$ with the commutation of the j -hat and i -hat cosines with the k -hat basis vector by

dropping the cosine dependence on $\hat{j} \times$ and $\hat{i} \times$ (because they have then operated on $\hat{k}$) giving the scale invariant, scalar version of cosine to the right of $\hat{k}$. In summary,

cosk → cos	after operating on $\hat{i}$ -hat or $\hat{j}$ -hat	cosk → 1 after operating on $\hat{k}$ -hat ($\hat{k}$ -hat rotational eigenvalue)
cosj → cos	after operating on $\hat{k}$ -hat or $\hat{i}$ -hat	cosj → 1 after operating on $\hat{j}$ -hat ($\hat{j}$ -hat rotational eigenvalue)
cosi → cos	after operating on $\hat{j}$ -hat or $\hat{k}$ -hat	cosi → 1 after operating on $\hat{i}$ -hat ($\hat{i}$ -hat rotational eigenvalue)

By defining what we have known as complex numbers as basis curl operators, the vector basis Euler's Theorem has been developed and demonstrated conclusively for **expk**. Similarly as for the basis sines and cosines, there are then three different forms of complex exponential $e^{\theta \hat{k} \times} = \exp(\theta \hat{k} \times (\cdot))$, $e^{\theta \hat{j} \times} = \exp(\theta \hat{j} \times (\cdot))$, or $e^{\theta \hat{i} \times} = \exp(\theta \hat{i} \times (\cdot))$, from which the sines and cosines are defined as above.

These are the $\hat{k}$ -hat exponential **expk**, the $\hat{j}$ -hat exponential **expj**, and the $\hat{i}$ -hat exponential **expi**, and there are hence three different forms of the vector basis form of Euler's formula, one for each these rotation operators around the three R^3 axes and hence rotation operators around each of the three standard basis vectors once appended with $\times$. The Euler forms are

$$\begin{aligned}\exp(\theta \hat{k} \times (\cdot)) &= e^{\theta \hat{k} \times} = \text{cosk } \theta + (\text{sink } \theta) \hat{k} \times \\ \exp(\theta \hat{j} \times (\cdot)) &= e^{\theta \hat{j} \times} = \text{cosj } \theta + (\text{sinj } \theta) \hat{j} \times \\ \exp(\theta \hat{i} \times (\cdot)) &= e^{\theta \hat{i} \times} = \text{cosi } \theta + (\text{sini } \theta) \hat{i} \times\end{aligned}$$

where, as for all quantities, must be fully specified in the 3-dimensional vector space (R^3 volume), and each of these must be specified with respect to a direction (say, either $\hat{i}$ and $\hat{j}$, for $\hat{k} \times$) whilst that for which it depends, the rotation axis, degenerates to the unitary eigenvalue and eigenvector when the operator operates on that basis vector. This explains why complex exponentials are circles, as are rotation matrices, as well as explaining their equivalence.

Say $e^{\theta \hat{k} \times} = \exp(\theta \hat{k} \times (\cdot))$ is specified in the x -direction with $\hat{i}$, then

$$e^{\theta \hat{k} \times} \hat{i} = \exp(\theta \hat{k} \times (\cdot)) \hat{i} = (\text{cosk } \theta) \hat{i} + (\text{sink } \theta) \hat{k} \times \hat{i} = (\text{cosk } \theta) \hat{i} + (\text{sink } \theta) \hat{j} = \hat{i} (\cos \theta) + \hat{j} (\sin \theta)$$

which is a circle on the real x - y plane generated and spanned by $\hat{i}$ and $\hat{j}$. Here, the path of the position coordinates curl and hence circle around the z axis generated by $\hat{k}$ in the real x - y - z volume. The fact that this is real (and not complex) is very exciting because of what mathematics I and others will be able to do with this fact. Including the latent variable, this is

$$\hat{i} (\cos \theta) + \hat{j} (\sin \theta) + \hat{k} (0) \quad \text{circles}$$

which is a unit radius circle, centred at the origin, starting at 0° (so the $\hat{j}$ -hat component is zero). And since it is real, all of the tools from vector calculus are available, such as the line (path) integral for **circumference** (path length)

$$\oint |e^{\theta \hat{k} \times} \hat{i}| d\theta = \int_0^{2\pi} \sqrt{(d \cos \theta / d\theta)^2 + (d \sin \theta / d\theta)^2 + (d(1) / d\theta)^2} d\theta = \int_0^{2\pi} \sqrt{\sin^2 \theta + \cos^2 \theta + 0^2} d\theta = \int_0^{2\pi} d\theta = 2\pi$$

and Stokes theorem for **area**

$$\begin{aligned}
\frac{1}{2} \oint x dy - y dx &= \frac{1}{2} \int_0^{2\pi} \cos \theta d \sin \theta - \sin \theta d \cos \theta = \frac{1}{2} \int_0^{2\pi} (\cos \theta (d \sin \theta / d \theta) - \sin \theta (d \cos \theta / d \theta)) d \theta = \int_0^{2\pi} (\cos^2 \theta + \sin^2 \theta) d \theta \\
&= \frac{1}{2} \int_0^{2\pi} d \theta \\
&= \pi
\end{aligned}$$

When extended to a Gauss distributed basis, a $\hat{k}$ -exponential is a torus of Gauss density.

Also, note that it is possible to have a circle defined as x for the i -hat direction, which then maps to both x and y (to i -hat and j -hat), and to have a circle defined as y for the j -hat direction which also maps to x and y (as a phase shifted x). If these are two components of a vector and have the same amplitude, then the two components together (as the vector) are then another circle with a larger amplitude and a phase shift. This is the logic behind a rotation matrix which is derived and discussed in the topic of **coordinate systems** where it is shown to be equivalent to the complex exponential (for each of the three axes). Note that the circle defined for the i -hat direction or the circle defined for the j -hat direction may either be zero (for example just an x component (i -hat) circle mapping to i -hat and j -hat) in which case it is an ordinary circle with no phase or amplitude change as above.

If both components of the defining vector are non-zero and the amplitudes are different, then the vector is an ellipse. If the frequencies are different then the vector is a Bowditch curve (Lissajous figure). For a common frequency, any individual component is a phasor.

Vector Basis de-Moivre's Theorem: $(\text{expk } \theta)^n = \text{cisk } n\theta$, $(\text{expj } \theta)^n = \text{cisj } n\theta$, and $(\text{expi } \theta)^n = \text{cisi } n\theta$

This theory not surprising also extends to de-Moivre's theorem for powers, which I am calling the vector basis form of de-Moivre's theorem. There are consequently three applications of the vector basis form of de-Moivre's theorem in a real R^3 vector volume for the standard basis, one for each of the three orthogonal rotational axes $\hat{k}$ -hat, $\hat{j}$ -hat, and $\hat{i}$ -hat:

$$\begin{aligned}
\exp(n\theta \hat{k} \times (\cdot)) &= e^{n\theta \hat{k} \times} = (\text{cisk } \theta + (\text{sink } \theta) \hat{k} \times)^n = \text{cisk } n\theta + (\text{sink } n\theta) \hat{k} \times \\
\exp(n\theta \hat{j} \times (\cdot)) &= e^{n\theta \hat{j} \times} = (\text{cosj } \theta + (\text{sinj } \theta) \hat{j} \times)^n = \text{cosj } n\theta + (\text{sinj } n\theta) \hat{j} \times \\
\exp(n\theta \hat{i} \times (\cdot)) &= e^{n\theta \hat{i} \times} = (\text{cosi } \theta + (\text{sini } \theta) \hat{i} \times)^n = \text{cosi } n\theta + (\text{sini } n\theta) \hat{i} \times
\end{aligned}$$

which are similarly real quantities (in a real R^3 vector space) when a necessary direction is specified (such as a zero phase right handed system). This then arbitrarily scales the frequency of the circles which is almost beyond words in terms of importance since it is that upon which all of spectroscopy and radio-tuning is based. A Fourier spectrum is essentially a scanning search for such circles, that is, for $\hat{k}$ -hat exponentials.

For the earlier circle for $e^{\theta \hat{k} \times} = \exp(\theta \hat{k} \times (\cdot))$ raised to a power n and pointing in the x -direction with $\hat{i}$

$$e^{n\theta \hat{k} \times} \hat{i} = \exp(n\theta \hat{k} \times (\cdot)) \hat{i} = (\text{cisk } n\theta) \hat{i} + (\text{sink } n\theta) \hat{k} \times \hat{i} = (\text{cisk } n\theta) \hat{i} + (\text{sink } n\theta) \hat{j} = \hat{i} (\cos n\theta) + \hat{j} (\sin n\theta)$$

which is the same circle that orbits the z axis n times as frequently over a given period of time (n times the frequency). You just watch and see how useful this makes this mathematics. Good times ahead. Especially so in the context of calculus, and quite possibly also for applications such as x-ray crystallography and other related scientific and engineering fields.

While I was looking for a sphere, I got a circle. So as a little aside here is proof of De-Moivre's theorem by mathematical induction which resulted while searching for spheres (and placed into this section from that of the **Riemann Sphere** which is next):

$$\begin{aligned}
 \left(e^{\theta \hat{k} \times}\right)^n \left(e^{\theta \hat{k} \times}\right)^m (\cdot) &= \left(e^{n\theta \hat{k} \times}\right) \left(e^{m\theta \hat{k} \times}\right) (\cdot) \\
 &= (\mathbf{cosk} \, m\theta + (\mathbf{sink} \, m\theta) \hat{k} \times) (\mathbf{cosk} \, n\theta + (\mathbf{sink} \, n\theta) \hat{k} \times) (\cdot) \\
 &= \mathbf{cosk} \, m\theta \left((\mathbf{cosk} \, n\theta) + (\mathbf{sink} \, n\theta) \hat{k} \times \right) + \left((\mathbf{sink} \, m\theta) \hat{k} \times \right) \left((\mathbf{cosk} \, n\theta) + (\mathbf{sink} \, n\theta) \hat{k} \times \right) \\
 &= (\mathbf{cosk} \, m\theta)(\mathbf{cosk} \, n\theta) + (\mathbf{cosk} \, m\theta)(\mathbf{sink} \, n\theta) \hat{k} \times + (\mathbf{sink} \, m\theta)(\mathbf{cosk} \, n\theta) \hat{k} \times + (\mathbf{sink} \, m\theta)(\mathbf{sink} \, n\theta) \hat{k} \times \hat{k} \times \\
 &= (\mathbf{cosk} \, m\theta)(\mathbf{cosk} \, n\theta) + (\mathbf{cosk} \, m\theta)(\mathbf{sink} \, n\theta) \hat{k} \times + (\mathbf{sink} \, m\theta)(\mathbf{cosk} \, n\theta) \hat{k} \times - (\mathbf{sink} \, m\theta)(\mathbf{sink} \, n\theta) \\
 &= ((\mathbf{cosk} \, m\theta)(\mathbf{cosk} \, n\theta) - (\mathbf{sink} \, m\theta)(\mathbf{sink} \, n\theta)) + ((\mathbf{cosk} \, m\theta)(\mathbf{sink} \, n\theta) + (\mathbf{sink} \, m\theta)(\mathbf{cosk} \, n\theta)) \hat{k} \times \\
 &= \mathbf{cosk} \, (m+n)\theta + (\mathbf{sink} \, (m+n)\theta) \hat{k} \times \\
 &= \exp((n+m)\theta \hat{k} \times) (\cdot) \\
 &= e^{(n+m)\theta \hat{k} \times} (\cdot)
 \end{aligned}$$

so de-Moivre's theorem is assumed true for n and true for m , hence it has been proved for $m+n$ for arbitrary m and n for the k -hat rotation axis with $i = \sqrt{-1}$. Note that this mathematics is equivalent to complex algebra and has only worked without assigning a basis direction in the first line of working because it is a circle multiplied by another circle on the same plane (around the k -hat axis and hence all in k -cross). This is because of the distribution on the 3rd and 4th line and commutation of k -cross with **cosk** and **sink** which otherwise is not valid (but is here, and is why 'complex numbers' appear to numbers when they are not numbers but rather operators) which I will show in the logic of the Riemann Sphere (which is now).

Riemann Sphere

Spheres are hard to obtain by ordinary means because in many cases one axis can't spin simultaneously with another as per finite rotations. This is noted often in quantum mechanics. I am thinking **Rubik's cubes and gyroscopes**, and the fact that you can't make an R move and a U move on a Rubiks cube at the same time [even with cubes that avoid lock-ups and can corner cut]. I say this because this may cause problems with drawing a circle on azimuth (with a k -hat exponential) and then rotating it as another circle on colatitude (with a j -hat exponential) to generate a sphere, which is what I am trying to do here. I think I have nevertheless succeeded with this.

I started working this out before I started with the **cosk**, **cosi**, and **cosj** notation, and it is the essential the reason why I started using this notation and found that there were three sines and three cosines (as operators of the three basis curls), in addition to the scalar sines and cosines, for the Riemann sphere. Obtaining the Riemann sphere as a function of basis curls [formally thought of as complex numbers] would otherwise not have been possible. Here is the early working with which I have tried to convey how I worked it out despite difficulties and doubts whether it could be worked out with $i = \hat{k} \times$ and $\hat{j} \times$ (the components of the standard basis appended with $\times$):

$$\begin{aligned}
\left(e^{\theta \hat{j} \times}\right)^m \left(e^{\theta \hat{k} \times}\right)^n \hat{i} &= \exp(m\theta \hat{j} \times) \exp(n\theta \hat{k} \times) \hat{i} \\
&= (\cos m\theta + (\sin m\theta) \hat{j} \times) (\cos n\theta + (\sin n\theta) \hat{k} \times) \hat{i} \\
&= (\cos m\theta + (\sin m\theta) \hat{j} \times) ((\cos n\theta) \hat{i} + (\sin n\theta) \hat{j}) \\
&= \cos m\theta ((\cos n\theta) \hat{i} + (\sin n\theta) \hat{j}) + (\sin m\theta) \hat{j} \times ((\cos n\theta) \hat{i} + (\sin n\theta) \hat{j}) \\
&= ((\cos m\theta)(\cos n\theta) \hat{i} + (\cos m\theta)(\sin n\theta) \hat{j}) + ((\sin m\theta)(\cos n\theta)(-\hat{k}) + (\sin m\theta)(\sin n\theta)(0)) \\
&= (\cos m\theta)(\cos n\theta) \hat{i} + (\cos m\theta)(\sin n\theta) \hat{j} - (\sin m\theta) \underline{(\cos n\theta)} \hat{k}
\end{aligned}$$

which I noted would be a sphere if it wasn't for the underlined factor in the $\hat{k}$ -hat component. So when I have interacted de-Moivre's theorem (the $\hat{k}$ -hat exponential for azimuth with the $\hat{j}$ -hat exponential for colatitude) to try to get a sphere I was disappointed because of the one term that was out (from what is known to be a spherical coordinate system), the underlined cosine (first problem – not an orthogonal circularly rotated circle). This was disappointing but I kept on trying to devise a way to overcome this. I noted that this is for the $\hat{k} \times$ cosine associated with the $\hat{k}$ direction. I then realized that this cosine (the $\hat{k} \times$ cosine) coupled with its own direction is equal to its direction, which was deduced after expanding it in power series to determine exactly what was going on (first problem solved).

$$\underline{(\cos n\theta)} \hat{k} = \cosh(n\theta \hat{k} \times (\cdot)) \hat{k} = \cosh(n\theta \hat{k} \times) \hat{k} = \hat{k}$$

and in the new notation this is $\underline{(\cos n\theta)} \hat{k} = (\mathbf{cosk}(n\theta)) \hat{k} = \hat{k}$ rather than with cosh

$$\begin{aligned}
\left(e^{\theta \hat{j} \times}\right)^m \left(e^{\theta \hat{k} \times}\right)^n \hat{i} &= \exp(m\theta \hat{j} \times) \exp(n\theta \hat{k} \times) \hat{i} \\
&= (\cos m\theta + (\sin m\theta) \hat{j} \times) (\underline{(\cos n\theta)} \hat{i} + (\sin n\theta) \hat{j}) \\
&= (\cos m\theta)(\cos n\theta) \hat{i} + (\cos m\theta)(\sin n\theta) \hat{j} - (\sin m\theta) \hat{k}
\end{aligned}$$

After initial excitement, I then realized that it was still not right. Note that I was tracking which sinusoids were $\hat{j}$ -hat dependent and which ones were $\hat{k}$ -hat dependent by reference to the frequency scalings m and n . The $(\cos n\theta)$ had already operated on the basis vector $\hat{k}$ -hat. If that had degenerated, then **that would have meant that there was a $\hat{j}$ -hat dependent harmonic in the $\hat{j}$ -hat component that would degenerated also (from the $\hat{j}$ -hat exponential), which would then have meant that it was no longer a sphere** (second problem – which harmonic degenerates?). If the underlined cosine degenerated, then I knew that the $(\mathbf{cosj}m\theta)$ in the $\hat{j}$ -hat components would have to degenerate also, and revert to its basis direction. But both of them degenerating was no longer spherical. Perhaps, I thought, that one degenerated and the other didn't? This was conjectural at the time [late September 2021] and so here came the need to define the **sink**, **sinj**, and **cosk**, **cosj**, rather than trying to remember [whilst actively doing the maths] which was associated with m and which with n , i.e., whether the $\hat{k} \times$ was in the cosine with the m or in the cosine with the n , and similarly for the $\hat{j} \times$ when expanding these functions in power series.

$\underline{(\cos n\theta)} \hat{k} = \hat{k} \underline{(\cos n\theta)}$ and here was the exact place where I found the need to define the **cosj** and **cosk**, and this next line of working I realized the need to order it like a manufacturing sausage train (I like manufacturing engineering documentaries and courses)

$(\cos j m \theta)(\sin n \theta) \hat{j} = (\cos j m \theta) \hat{j}(\sin n \theta)$ since the sine has operated on its basis vector but the cosine not as yet so

$(\cos j m \theta)(\sin n \theta) \hat{j} = (\cos j m \theta) \hat{j}(\sin n \theta) = \hat{j}(\sin n \theta)$ where $(\cos j m \theta) \hat{j} = \hat{j}$ so that

$$\begin{aligned} \left(e^{\theta \hat{j} \times}\right)^m \left(e^{\theta \hat{k} \times}\right)^n \hat{i} &= \exp\left(\theta \hat{j} \times (\cdot)\right)^m \exp\left(\theta \hat{k} \times (\cdot)\right)^n \hat{i} = \left(\cos j m \theta + (\sin j m \theta) \hat{j} \times\right) \left(\cos k n \theta + (\sin k n \theta) \hat{k} \times\right) \hat{i} \\ &= (\cos j m \theta)(\cos n \theta) \hat{i} + \underline{(\cos j m \theta)(\sin n \theta) \hat{j}} - (\sin j m \theta)(\cos n \theta) \hat{k} \\ &= (\cos j m \theta) \hat{i} (\cos n \theta) + \underline{(\cos j m \theta) \hat{j} (\sin n \theta)} - (\sin j m \theta) \hat{k} (\cos n \theta) \\ &= \hat{i} (\cos m \theta) (\cos n \theta) + \hat{j} (\sin n \theta) - \hat{k} (\sin m \theta) (\cos n \theta) \end{aligned}$$

so it was actually the j -hat cosine which degenerates to its basis vector j -hat, not the k -hat cosine (second problem solved). Thankfully they don't both degenerate. The k -hat cosine doesn't degenerate because it is converted to a scalar cosine by operating on i -hat, and hence doesn't degenerate later (since it is then scalar) when the i -hat is converted to k -hat by j -cross ($j \times$).

So the **coupled exponentials** are **polar sphericals** after all: azimuth on i -hat and k -hat (with frequency m), and colatitude on j -hat (with frequency n). So I thought this expression was (elated), then wasn't (first problem) then was (first problem solved), then wasn't (second problem), polar sphericals (with huge disappointment), but actually after much further resolving (second problem solved), is (polar sphericals). I think I have actually got it this time around (polar sphericals). Looking for **Harmony of the Spheres**. I still have uncertainty with this: rotations of spheres almost always occur with a single spin axis (the principle axis), and Rubik's cubes can spin on one single axis at any one time. This is probably the physics of why gyroscopes work, as per a right-handed Maxwell basis with discrete in the very small. Nevertheless, spherical coordinate systems occur all the time including especially in Maxwell theory. I'm here thinking **ball and socket**, rather than **wheel and axel**.

Defining the **k -hat cosine** (**cosk**), **j -hat cosine** (**cosj**), **k -hat sine** (**sink**), and **j -hat sine** (**sinj**) for the complete working, which I think I have now got rock solid (with the new notation, sorting which are operators and which are scalars, and the commutation order and properties of the operators). What I will be doing from here on in, and retrospectively with earlier parts of this document [as a reader may have noticed if reading from the beginning (where I made the new definitions)], is keeping the basis vectors to the left of those which depend on the basis vector curls (those that depend on $k \times$, $j \times$, and $i \times$), and hence are operators, and those which are independent of the basis curls (have been operated on already) I will keep on the right. A **manufacturing sausage train from right to left**. I had to update the working for the k -hat exponential above. Here is the working as I have found works for the sines and cosines and polar sphericals (from my generalization of Euler's theorem to the Riemann sphere)

$$\begin{aligned}
\left(e^{\theta \hat{j} \times}\right)^m \left(e^{\theta \hat{k} \times}\right)^n \hat{i} &= \exp\left(\theta \hat{j} \times (\cdot)\right)^m \exp\left(\theta \hat{k} \times (\cdot)\right)^n \hat{i} \\
&= \exp\left(m\theta \hat{j} \times (\cdot)\right) \exp\left(n\theta \hat{k} \times (\cdot)\right) \hat{i} \\
&= \left(\cos j m\theta + (\sin j m\theta) \hat{j} \times\right) \left(\cos k n\theta + (\sin k n\theta) \hat{k} \times\right) \hat{i} \\
&= \left(\cos j m\theta + (\sin j m\theta) \hat{j} \times\right) \left((\cos k n\theta) \hat{i} + (\sin k n\theta) \hat{k} \times \hat{i}\right) \\
&= \left(\cos j m\theta + (\sin j m\theta) \hat{j} \times\right) \left(\hat{i}(\cos n\theta) + \hat{j}(\sin n\theta)\right) \\
&= (\cos j m\theta) \left(\hat{i}(\cos n\theta) + \hat{j}(\sin n\theta)\right) + (\sin j m\theta) \hat{j} \times \left(\hat{i}(\cos n\theta) + \hat{j}(\sin n\theta)\right) \\
&= \left((\cos j m\theta) \hat{i}(\cos n\theta) + (\cos j m\theta) \hat{j}(\sin n\theta)\right) + \left((\sin j m\theta)(\hat{j} \times \hat{i})(\cos n\theta) + (\sin j m\theta)(\hat{j} \times \hat{j})(\sin n\theta)\right) \\
&= \left(\hat{i}(\cos m\theta)(\cos n\theta) + \hat{j}(\sin m\theta)(\cos n\theta)\right) + \left((\sin j m\theta)(-\hat{k})(\cos n\theta) + (\sin j m\theta)(0)(\sin n\theta)\right) \\
&= \hat{i}(\cos m\theta)(\cos n\theta) + \hat{j}(\sin m\theta)(\cos n\theta) - \hat{k}(\sin m\theta)(\sin n\theta)
\end{aligned}$$

that is

$$\left(e^{\theta \hat{j} \times}\right)^m \left(e^{\theta \hat{k} \times}\right)^n \hat{i} = \hat{i}(\cos m\theta)(\cos n\theta) + \hat{j}(\sin m\theta)(\cos n\theta) - \hat{k}(\sin m\theta)(\sin n\theta)$$

which is definitively a sphere with the requisite x-component, y-component, and z-component with radial frequency of m in azimuth and n in colatitude. Best of all, is entirely real, with the real R^3 volumetric space independently of a non-existent complex field **so there is no uncertainty as to when it is and when it is not applicable**, but with the extension of all of the same theory that complex numbers provide with Euler's theorem and de-Moivre's theorem. Overwhelmingly pleased with this. This shows when it is possible to scale the azimuthal and colatitude frequencies [although not together as they do not commute], similarly as circles on the plane. **Harmony of the Spheres.** That's what I have been looking for.

Yep, now rock solid. This is with the theory for sines and cosines in terms of the correct basis curls, single component basis cross products; **sink** and **cosk**, **sinj** and **cosj**, and **sini** and **cosi**. This will then be a spherical coordinate system (Riemann sphere) with arbitrary frequencies on azimuth and colatitude. This is a generalization of Euler's and de-Moivre's theorems for complex exponentials and powers, but for the product of the $\hat{j}$ -hat and $\hat{k}$ -hat exponentials which are real vector quantities. I am wanting to model these powers (m and n) as energy levels of hydrogen (Balmer's lines). In my opinion, the energy levels of hydrogen was the real idea behind Planck's hypothesis (although Planck forgot to mention this), i.e., the Balmer lines, which Balmer showed are a **discrete** $1/n^2$ law in analogy to gravity and electromagnetism as **continuous** $1/R^2$ laws.

As a caveat, this is 'sort of' a generalization of the de-Moivres theorem which generalizes from circles to spheres. I say sort of because it is not quite so simple because for the fact that the $\hat{k}$ -hat and $\hat{j}$ -hat exponentials don't commute and hence the coupling of the $\hat{k}$ -hat exponential and $\hat{j}$ -hat exponential can't be arbitrarily raised to powers together (essentially the same problem in matrices of not being diagonal). They have to be raised to powers individually before being multiplied together and assigned a basis direction. I am hoping that this will sort itself out when I start doing calculus (the diffusion calculus of arbitrary order). Turns out it is relevant for nilpotent operators. Since it doesn't quite work like de-Moivre's theorem because of non-commutativity of the $\hat{j}$ -hat and $\hat{k}$ -hat exponentials, and hence no simplifications on raising to powers, it is more like a generalization to Euler's theorem. The problems I am having with this, I think, relate to the eighth and ninth base SI units radians and steradians, units for angles and solid angles. Still working it out. Think that the coupling of the $\hat{j}$ -hat exponential and the $\hat{k}$ -hat exponential might need to be defined in terms of an

integral, so as to produce a solid angle (steradian) of a sphere. Don't know yet. I am planning on having a further document after this one applying this theory to a full calculus (of arbitrary order).

This is the real (rather than complex) equivalent on R^3 of the generalization of the **Argand diagram** (Gauss plane) to the **Riemann sphere** where now all quantities are real in the vectors space R^3 with $\times$ applied to the standard basis. This is my generalization, based on known-to-be-good mathematics, of de-Moivres theorem for complex exponentials which generates the Riemann sphere [with the caveat of non-commutativity], and was my main goal with this current work, that is, getting everything real (in R^3) and spheres (from circles). The spherical coordinate system is featured in all electromagnetic theory including all undergraduate and postgraduate texts so I know that this is useful mathematics, and the complex exponentials for each axis (**expk**, **expj**, and **expi**) is next demonstrated to be the rotation matrices (R_z , R_y , and R_x).

The **Riemann sphere** is the **orthographic projection** of a **circle on the complex plane into a complex sphere**, the projection occurring through the straight lines tracing out from the complex circle. From earlier in the document, it is now here known that the complex plane is actually the real x-y plane and, as has been shown and proved here, the stereographic projection of the circle from this plane around the $\hat{k}$ -axis (the z-axis) then resulted in the Riemann sphere in a real volume in R^3 (rather than some extended complex field) [note that the way it has been parameterized (the $\hat{j}$ -hat exponential times the $\hat{k}$ -hat exponential) it is actually the $\hat{j}$ -hat axis and x-z plane]. This very significant outcome has resulted from the earlier results that I kept noting as interesting, those whereby the harmonics in their own direction degenerated. This functional degeneration from a harmonic was **(1)** to the basis variable itself in the case of $\hat{k}$ -hat cosine and **(2)** a linear scaling of the basis variable in the case of $\hat{k}$ -hat sine (the degenerated sine not relevant here but is to other contexts). These essentially resulted from the small angle approximations in $k \times$ and $j \times$. These are the straight lines of the stereographic projection from the plane circle to a sphere. For the $\hat{j}$ -hat direction

$$\begin{aligned}
 \left(e^{\theta \hat{j} \times}\right)^m \left(e^{\theta \hat{k} \times}\right)^n \hat{j} &= \exp\left(\theta \hat{j} \times (\cdot)\right)^m \exp\left(\theta \hat{k} \times (\cdot)\right)^n \hat{j} \\
 &= \exp\left(m\theta \hat{j} \times (\cdot)\right) \exp\left(n\theta \hat{k} \times (\cdot)\right) \hat{j} \\
 &= \left(\cos j m \theta + (\sin j m \theta) \hat{j} \times\right) \left(\cos k n \theta + (\sin k n \theta) \hat{k} \times\right) \hat{j} \\
 &= \left(\cos j m \theta + (\sin j m \theta) \hat{j} \times\right) \left((\cos k n \theta) \hat{j} + (\sin k n \theta) \hat{k} \times \hat{j}\right) \\
 &= \left(\cos j m \theta + (\sin j m \theta) \hat{j} \times\right) \left(\hat{j} (\cos n \theta) - \hat{i} (\sin n \theta)\right) \\
 &= (\cos j m \theta) \left(\hat{j} (\cos n \theta) - \hat{i} (\sin n \theta)\right) + (\sin j m \theta) \hat{j} \times \left(\hat{j} (\cos n \theta) - \hat{i} (\sin n \theta)\right) \\
 &= \left((\cos j m \theta) \hat{j} (\cos n \theta) - (\cos j m \theta) \hat{i} (\sin n \theta)\right) + \left((\sin j m \theta) \hat{j} \cancel{\times} \hat{j} (\cos n \theta) - (\sin j m \theta) \hat{j} \times \hat{i} (\sin n \theta)\right) \\
 &= -\hat{i} (\cos m \theta) (\sin n \theta) + \hat{j} (\cos n \theta) + \hat{k} (\sin m \theta) (\sin n \theta)
 \end{aligned}$$

In the $\hat{k}$ -hat direction

$$\begin{aligned}
\left(e^{\theta \hat{j} \times}\right)^m \left(e^{\theta \hat{k} \times}\right)^n \hat{k} &= \exp\left(m\theta \hat{j} \times (\cdot)\right) \exp\left(n\theta \hat{k} \times (\cdot)\right) \hat{k} \\
&= \left(\cos j m\theta + (\sin j m\theta) \hat{j} \times\right) \left(\cos k n\theta + (\sin k n\theta) \hat{k} \times\right) \hat{k} \\
&= \left(\cos j m\theta + (\sin j m\theta) \hat{j} \times\right) \left((\cos k n\theta) \hat{k} + (\sin k n\theta) \hat{k} \times \hat{k}\right) \\
&= \left(\cos j m\theta + (\sin j m\theta) \hat{j} \times\right) \left(\hat{k}\right) \\
&= \left((\cos j m\theta) \hat{k} + (\sin j m\theta) \hat{j} \times \hat{k}\right) \\
&= \hat{i} (\sin m\theta) + \hat{k} (\cos m\theta)
\end{aligned}$$

which is the $\hat{j}$ -hat circle (around the y -axis on the x - z plane) from which the stereographic projection takes place into the sphere. In summary, the components of the Riemann sphere, with the azimuthal and colatitude components raised to powers by de-Moivre's theorem, are

$$\begin{aligned}
\left(e^{\theta \hat{j} \times}\right)^m \left(e^{\theta \hat{k} \times}\right)^n \hat{i} &= \hat{i} (\cos m\theta) (\cos n\theta) + \hat{j} (\sin n\theta) - \hat{k} (\sin m\theta) (\cos n\theta) \\
\left(e^{\theta \hat{j} \times}\right)^m \left(e^{\theta \hat{k} \times}\right)^n \hat{j} &= -\hat{i} (\cos m\theta) (\sin n\theta) + \hat{j} (\cos n\theta) + \hat{k} (\sin m\theta) (\sin n\theta) \\
\left(e^{\theta \hat{j} \times}\right)^m \left(e^{\theta \hat{k} \times}\right)^n \hat{k} &= \hat{i} (\sin m\theta) + \hat{k} (\cos m\theta)
\end{aligned}$$

Coordinate Systems - Cylindrical

From the discussion of the $\hat{k}$ -hat exponential, the components of the $\hat{k}$ -hat exponential when it is specified in the three basis directions are

$$\begin{aligned}
e^{\theta \hat{k} \times} \hat{i} &= \exp\left(\theta \hat{k} \times (\cdot)\right) \hat{i} = \hat{i} (\cos \theta) + \hat{j} (\sin \theta) + \hat{k}(0) \\
e^{\theta \hat{k} \times} \hat{j} &= \exp\left(\theta \hat{k} \times (\cdot)\right) \hat{j} = -\hat{i} (\sin \theta) + \hat{j} (\cos \theta) + \hat{k}(0) \\
e^{\theta \hat{k} \times} \hat{k} &= \exp\left(\theta \hat{k} \times (\cdot)\right) \hat{k} = \hat{i}(0) + \hat{j}(0) + \hat{k}(1)
\end{aligned}$$

where I noted that the $\hat{k}$ -hat exponential defined in the $\hat{k}$ -hat direction degenerated to its own basis variable ($\hat{k}$ -hat) and further noted that this was the **unitary eigenvalue** of a rotation. Set out in this way, it is clear (for those familiar with linear algebra) that this has the component form of the **rotation matrix**. Hence, when defined in the context of a basis $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, which is a linear combination of the basis-hats (scaled by the component scalars), then the $\hat{k}$ -hat exponential can be expressed as

$$\begin{aligned}
e^{\theta \hat{k} \times} \vec{r} &= e^{\theta \hat{k} \times} (x\hat{i} + y\hat{j} + z\hat{k}) \\
&= e^{\theta \hat{k} \times} x\hat{i} + e^{\theta \hat{k} \times} y\hat{j} + e^{\theta \hat{k} \times} z\hat{k} \\
&= x \left(\cos k \theta + (\sin k \theta) \hat{k} \times \right) \hat{i} + y \left(\cos k \theta + (\sin k \theta) \hat{k} \times \right) \hat{j} + z \left(\cos k \theta + (\sin k \theta) \hat{k} \times \right) \hat{k} \\
&= x \left(\hat{i} \cos \theta + \hat{j} \sin \theta + \hat{k} 0 \right) + y \left(\hat{j} \cos \theta - \hat{i} \sin \theta + \hat{k} 0 \right) + z \left(\hat{i} 0 + \hat{j} 0 + \hat{k} 1 \right) \\
&= \hat{i} (x \cos \theta - y \sin \theta) + \hat{j} (x \sin \theta + y \cos \theta) + \hat{k} z
\end{aligned}$$

the last line of which being the x-y components of a rotation of the x-y plane by a rotation matrix and hence can intuitively (by someone familiar with linear algebra) be expressed as

$$e^{\theta \hat{k} \times} \vec{r} = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \vec{r}$$

This shows why the degeneration of the k -hat exponential in the k -hat direction is the **rotation eigenvalue**.

This means that the k -hat exponential defined (as an operator) in the context of $k \times$, rather than a scalar in $i = \sqrt{-1}$ (which is really just a updated interpretation of the same thing, notably real rather than complex, and operator rather than scalar or number), **the k -hat exponential (which is equivalent to the complex exponential) is the rotation matrix** when coupled with a basis, the basis being the i -hat, j -hat, and k -hat components. This is how it arises, from basis curls [single component basis cross products with the properties of $\sqrt{-1}$ in the respective directions]. Here is the full workings in terms of **column basis**, **row basis**, and **matrix basis** whereby the **column basis** and **row basis** is expanded into a **matrix basis** as a system of **diadics** (diadic basis vector expansion to a matrix basis).

Theorem: The k -hat exponential **expk** is equivalent to the complex exponential with $i = \hat{k} \times$.

Theorem: The k -hat exponential **expk** is equivalent to the x-y plane rotation matrix.

Corollary: The complex exponential is one of the three rotation matrices, depending on which one of the three R^3 planes for which the complex plane is defined to correspond, **expk** if corresponding to the x-y plane

Theorem: The j -hat and i -hat isometries for **expk**, **expj** and **expi**, form the complete congruent set of R^3 rotation matrices with **expk**.

Note that this is with my interpretation of diadics whereby I prime the different dimensions (I have also spelt diadic rather than dyadic): I have been working with i -hat, j -hat, and k -hat as vectors so the row or column orientation hasn't mattered because it was 'either or'. Either a row or a column, it didn't matter. Now it does matter because vectors are expanding into matrices by diadics so I have a decision; are columns the primed basis components or are the rows? I'm choosing i -hat, j -hat, and k -hat (unprimed) to be column vectors and the primes i' -hat, j' -hat, and k' -hat, to be row vectors [although I prefer columns to be the primed components because the primes look like columns so it is easier to remember but then I would have to go back over the entire document changing all the notation and the primes would then include an enormous amount of clutter that would detract from the arguments regarding complex numbers - so that's my reasoning; also unprimed vectors are generally defined as column vectors at UNSW]. Taking the unprimed vector basis as columns, the primed vector basis as rows, the x-component of the **row vector basis** is $\hat{i}' = (1, 0, 0)$ and the x-component of the **column vector basis** is $\hat{i} = (1, 0, 0)^T$ which do not commute but are associative. The row basis provides a vector of ones for each of the positions of the row vector and the column basis provides a column vector of ones for each of the positions of the column vector. By diadics, the x-x component of the matrix (outer product) basis is then

$$\hat{i} \hat{i}' = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} (1, 0, 0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and the alternate non-commutative ordering of the components (inner product) is scalar unitary $\hat{i}'\hat{i} = (1, 0, 0) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 1$

Similarly for the y-component and z-component of the row basis, column basis, and y-y and z-z matrix basis, and each of the remaining elements of the matrix basis. Hence $\hat{i}'\hat{i} = \hat{j}'\hat{j} = \hat{k}'\hat{k} = 1$. For cross diadics, the inner products are zero scalar so $\hat{i}'\hat{j} = \hat{i}'\hat{k} = \hat{j}'\hat{i} = \hat{j}'\hat{k} = \hat{k}'\hat{i} = \hat{k}'\hat{j} = 0$. For example

$$\hat{i}'\hat{j} = (1, 0, 0) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 0 \quad \text{and} \quad \hat{i}'\hat{k} = (1, 0, 0) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0$$

The basis outer products populate the rest of the $3 \times 3 = 9$ components of the matrix basis, for example $\hat{i}\hat{j}' = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} (0, 1, 0) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Hence, the diadic expansion of the column and row vector basis by outer product then produces the matrix basis which provides a square system of ones for each of the positions for the 3×3 square. In full, the **vector row basis** is

$$\hat{i}' = (1, 0, 0) \quad \hat{j}' = (0, 1, 0) \quad \hat{k}' = (1, 0, 0)$$

The vector column basis is

$$\hat{i} = (1, 0, 0)' \quad \hat{j} = (0, 1, 0)' \quad \hat{k} = (1, 0, 0)'$$

And the **matrix basis**, from diadically expanding the vector column basis with the vector row basis by outer product is

$$\hat{i}\hat{i}' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \hat{i}\hat{j}' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \hat{i}\hat{k}' = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\hat{j}\hat{i}' = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \hat{j}\hat{j}' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \hat{j}\hat{k}' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\hat{k}\hat{i}' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \hat{k}\hat{j}' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \hat{k}\hat{k}' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\text{reference table} \begin{pmatrix} \hat{i}\hat{i}' & \hat{i}\hat{j}' & \hat{i}\hat{k}' \\ \hat{j}\hat{i}' & \hat{j}\hat{j}' & \hat{j}\hat{k}' \\ \hat{k}\hat{i}' & \hat{k}\hat{j}' & \hat{k}\hat{k}' \end{pmatrix}$$

noting that the actual diadic expansion into the rotation matrix pertains to a further column basis upon which the $\hat{k}$ -hat matrix operates. The **reference table** as to the positions of the components of a matrix basis within a matrix (which are otherwise latently behind-the-scenes generating the space when in matrix rather than component form) is included for reference next to the matrix

components above (which is somewhat confusing because of priming for rows rather than columns for the outer product, which I have earlier mentioned my reasoning for this, that is, why I have rows primed rather than columns). In symbolic algebra systems, what I would like is an option to ‘**make outer product matrix basis visible**’ and similarly with vectors ‘**make vector basis visible**’. This might be an opportunity for those that develop such software to extend the functionality of their wares. I know I would find it useful to ‘have a peek’ (similarly as pointers in c-programming) from time to time at what basis I am working in and which priming convention is adopted (whether unprimed components are column vectors or row vectors). Bases in mathematics are essentially memory addresses exactly as pointers in c-programming, which is that which makes c such a powerful programming language.

Hence, the diadic vector basis expansion of the k -hat exponential into the matrix basis proceeds as

$$\begin{aligned} e^{\theta \hat{k} \times} \vec{r} &= (\hat{i} \cos \theta + \hat{j} \sin \theta + \hat{k} 0)x + (-\hat{i} \sin \theta + \hat{j} \cos \theta + \hat{k} 0)y + (\hat{i} 0 + \hat{j} 0 + \hat{k} 1)z \\ &= (\hat{i} \cos \theta + \hat{j} \sin \theta + \hat{k} 0)(\hat{i}'\hat{i})x + (-\hat{i} \sin \theta + \hat{j} \cos \theta + \hat{k} 0)(\hat{j}'\hat{j})y + (\hat{i} 0 + \hat{j} 0 + \hat{k} 1)(\hat{k}'\hat{k})z \\ &= (\hat{i} \hat{i}' \cos \theta + \hat{j} \hat{j}' \sin \theta + \hat{k} \hat{i}' 0)(\hat{i} x) + (-\hat{i} \hat{j}' \sin \theta + \hat{j} \hat{j}' \cos \theta + \hat{k} \hat{j}' 0)(\hat{j} y) + (\hat{i} \hat{k}' 0 + \hat{j} \hat{k}' 0 + \hat{k} \hat{k}' 1)(\hat{k} z) \end{aligned}$$

that is

$$\begin{aligned} e^{\theta \hat{k} \times} \vec{r} &= \left((\hat{i} \cos \theta + \hat{j} \sin \theta + \hat{k} 0), (-\hat{i} \sin \theta + \hat{j} \cos \theta + \hat{k} 0), (\hat{i} 0 + \hat{j} 0 + \hat{k} 1) \right) \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ &= \left(\begin{pmatrix} \cos \theta \\ \sin \theta \\ 0 \end{pmatrix}, \begin{pmatrix} -\sin \theta \\ \cos \theta \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ &= R_z \vec{r} \end{aligned} \quad \text{that is, } e^{\theta \hat{k} \times} (\cdot) = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} (\cdot)$$

where $(\cdot)$ is a vector function and $R_z = R_z(\theta)$ is the **k -hat rotation matrix**. This is hence shown to be equivalent and synonymous with the **k -hat exponential, expk**, which was further shown to be equivalent to the **complex exponential** $e^{i\theta}$ from the relation $i = \hat{k} \times$. This is the same which turns a slice of a Rubiks cube. This is Euler’s formula for a complex exponential in its full glory. This is the complete elucidation of the formula for the k -hat exponential **expk** as a rotational operator of $\hat{k} \times$.

Hence, the k -hat exponential **expk** (what was formally known as the complex exponential but now known to be an operator and entirely real in a real vector volume) is the **standard counter-clockwise rotation** $\mathcal{U}$ for the x - y plane. It’s conjugate, the negative k -hat exponential is the **standard clockwise rotation** $\mathcal{V}$. Similarly for the x and z plane with the j -hat exponential (around the y axis) and for the y and z plane with the i -hat exponential (around the x axis). For the j -hat exponential **expj**, the components in the basis directions are

$$\begin{aligned} e^{\theta \hat{j} \times} \hat{i} &= \exp(\theta \hat{j} \times (\cdot)) \hat{i} = (\cos \theta + \sin \theta \hat{j} \times (\cdot)) \hat{i} = \hat{i} (\cos \theta) + \hat{j} (0) + \hat{k} (-\sin \theta) \\ e^{\theta \hat{j} \times} \hat{j} &= \exp(\theta \hat{j} \times (\cdot)) \hat{j} = (\cos \theta + \sin \theta \hat{j} \times (\cdot)) \hat{j} = \hat{i} (0) + \hat{j} (1) + \hat{k} (0) \\ e^{\theta \hat{j} \times} \hat{k} &= \exp(\theta \hat{j} \times (\cdot)) \hat{k} = (\cos \theta + \sin \theta \hat{j} \times (\cdot)) \hat{k} = \hat{i} (\cos \theta) + \hat{j} (0) + \hat{k} (\sin \theta) \end{aligned}$$

and coupled with the vector basis $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and expansion of the vector basis into a matrix basis, then the j -hat exponential **expj** is expressed as

$$e^{\theta \hat{j} \times} \vec{r} = \begin{pmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{pmatrix} \vec{r} = R_y \vec{r} \quad \text{that is} \quad e^{\theta \hat{j} \times} (\cdot) = \begin{pmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{pmatrix} (\cdot)$$

where $(\cdot)$ is a vector function and $R_y = R_y(\theta)$ is the j -hat rotation matrix, synonymously with the j -hat exponential **expj**. This rotates a vector around the y -axis, hence the unitary eigenvalue for the j -hat component (where the eigenvector is j -hat). For the i -hat exponential **expi**, the components in the basis directions are

$$\begin{aligned} e^{\theta \hat{i} \times} \hat{i} &= \exp(\theta \hat{i} \times (\cdot)) \hat{i} = (\cos \theta + \sin \theta \hat{i} \times (\cdot)) \hat{i} = \hat{i} (1) + \hat{j} (0) + \hat{k} (0) \\ e^{\theta \hat{i} \times} \hat{j} &= \exp(\theta \hat{i} \times (\cdot)) \hat{j} = (\cos \theta + \sin \theta \hat{i} \times (\cdot)) \hat{j} = \hat{i} (0) + \hat{j} (\cos \theta) + \hat{k} (\sin \theta) \\ e^{\theta \hat{i} \times} \hat{k} &= \exp(\theta \hat{i} \times (\cdot)) \hat{k} = (\cos \theta + \sin \theta \hat{i} \times (\cdot)) \hat{k} = \hat{i} (0) + \hat{j} (-\sin \theta) + \hat{k} (\cos \theta) \end{aligned}$$

and coupled with the vector basis $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and expansion of the vector basis into a matrix basis, then the i -hat exponential **expi** is expressed as

$$e^{\theta \hat{i} \times} \vec{r} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{pmatrix} \vec{r} = R_x \vec{r} \quad \text{that is} \quad e^{\theta \hat{i} \times} (\cdot) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{pmatrix} (\cdot)$$

where $(\cdot)$ is a vector function and $R_x = R_x(\theta)$ is the i -hat rotation matrix, synonymously with the i -hat exponential **expi**. This rotates a vector around the x -axis, hence the unitary eigenvalue for the i -hat component and the unit eigenvector for the unitary eigenvalue of the i -hat direction (the rotational axis).

de-Moivre's theorem for powers of the basis curl exponentials then modulates the rotations. For example, frequency scales for a continuous theta θ , or discrete curls for a fix radial unit of rotation, say $\theta = \pi / 2$ for a quarter rotation on a Rubiks cube for which then de-Moivre powers make the requisite number of quarter rotations. So here is another proof of de-Moivre's theorem: This is as simple as $R_z(n\theta)R_z(m\theta) = R_z(n\theta + m\theta) = R_z((n+m)\theta)$ when it is known that the rotation matrix is a k -hat exponential **expk**. For a full proof, from earlier with **expk** with argument $m\theta$ having operated on $\vec{r} = \hat{i}x + \hat{j}y + \hat{k}z$

$$e^{m\theta \hat{k} \times} \vec{r} = \hat{i}(x \cos m\theta - y \sin m\theta) + \hat{j}(x \sin m\theta + y \cos m\theta) + \hat{k}z$$

and left-to-right multiplying both sides by another k -hat exponential **expk** with argument $n\theta$, $e^{n\theta \hat{k} \times} = \cos n\theta + (\sin n\theta) \hat{k} \times$

$$\begin{aligned}
e^{n\theta\hat{k}\times} e^{m\theta\hat{k}\times} \vec{r} &= e^{n\theta\hat{k}\times} \left(e^{m\theta\hat{k}\times} \vec{r} \right) \\
&= e^{n\theta\hat{k}\times} \left(\hat{i}(x \cos m\theta - y \sin m\theta) + \hat{j}(x \sin m\theta + y \cos m\theta) + \hat{k}z \right) \\
&= \left(\text{cosk } n\theta + (\text{sink } n\theta)\hat{k} \times \right) \left(\hat{i}(x \cos m\theta - y \sin m\theta) + \hat{j}(x \sin m\theta + y \cos m\theta) + \hat{k}z \right) \\
&= (\text{cosk } n\theta) \left(\hat{i}(x \cos m\theta - y \sin m\theta) + \hat{j}(x \sin m\theta + y \cos m\theta) + \hat{k}z \right) \\
&\quad + \left((\text{sink } n\theta)\hat{k} \times \right) \left(\hat{i}(x \cos m\theta - y \sin m\theta) + \hat{j}(x \sin m\theta + y \cos m\theta) + \hat{k}z \right) \\
&= \left((\text{cosk } n\theta)\hat{i}(x \cos m\theta - y \sin m\theta) + (\text{cosk } n\theta)\hat{j}(x \sin m\theta + y \cos m\theta) + (\text{cosk } n\theta)\hat{k}z \right) \\
&\quad + \left((\text{sink } n\theta)\hat{k} \times \hat{i}(x \cos m\theta - y \sin m\theta) + (\text{sink } n\theta)\hat{k} \times \hat{j}(x \sin m\theta + y \cos m\theta) + (\text{sink } n\theta)\hat{k} \times \hat{k}z \right) \\
&= \left(\hat{i}(\cos n\theta)(x \cos m\theta - y \sin m\theta) + \hat{j}(\cos n\theta)(x \sin m\theta + y \cos m\theta) + \hat{k}z \right) \\
&\quad + \left((\text{sink } n\theta)\hat{j}(x \cos m\theta - y \sin m\theta) - (\text{sink } n\theta)\hat{i}(x \sin m\theta + y \cos m\theta) + (\text{sink } n\theta)\hat{k}(0)z \right) \\
&= \left(\hat{i}(\cos n\theta)(x \cos m\theta - y \sin m\theta) + \hat{j}(\cos n\theta)(x \sin m\theta + y \cos m\theta) + \hat{k}z \right) \\
&\quad + \left(\hat{j}(\sin n\theta)(x \cos m\theta - y \sin m\theta) - \hat{i}(\sin n\theta)(x \sin m\theta + y \cos m\theta) + \hat{k}0 \right) \\
&= \left(\hat{i}(x \cos n\theta \cos m\theta - y \cos n\theta \sin m\theta) + \hat{j}(x \cos n\theta \sin m\theta + y \cos n\theta \cos m\theta) + \hat{k}z \right) \\
&\quad + \left(\hat{j}(x \sin n\theta \cos m\theta - y \sin n\theta \sin m\theta) - \hat{i}(x \sin n\theta \sin m\theta + y \sin n\theta \cos m\theta) + \hat{k}0 \right) \\
&= \hat{i}(x \cos n\theta \cos m\theta - x \sin n\theta \sin m\theta - y \cos n\theta \sin m\theta - y \sin n\theta \cos m\theta) \\
&\quad + \hat{j}(x \cos n\theta \sin m\theta + x \sin n\theta \cos m\theta + y \cos n\theta \cos m\theta - y \sin n\theta \sin m\theta) + \hat{k}z \\
&= \hat{i}(x \cos(n+m)\theta - y \sin(n+m)\theta) + \hat{j}(x \sin(n+m)\theta + y \cos(n+m)\theta) + \hat{k}z \\
&= \begin{pmatrix} \cos(n+m)\theta & -\sin(n+m)\theta & 0 \\ \sin(n+m)\theta & \cos(n+m)\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\
&= e^{(n+m)\theta\hat{k}\times} \vec{r}
\end{aligned}$$

which is the rotation matrix for $m = 0, 1, 2, \dots$ and for $n = 0, 1, 2, \dots$ so is true for all m and n and hence it works exactly as it should and hence these claims in this document results in bona-fide known-to-be-good mathematical outcomes. Note also that since everything occurs in R^3 coordinate systems, coordinate systems must be overwhelmingly easy to come by. The k -hat exponential **expk** and the **expk** isometries **expj** and **expi**, in the context of de-Moivre's theorem and its generalization, is probably why.

The rotation matrices, and hence the basis exponentials (**expk**, **expj**, and **expi**), are also the cylindrical coordinate systems with the cylindrical axis in the basis-hat direction, i.e., in the direction of the unitary eigenvalue and its basis eigenvector (k -hat for example). For the k -hat exponential **expk**, the resulting coordinate system is cylindrical with cylindrical axis in the z direction, that is, the cylindrical coordinate system with cylindrical polar angle θ , cylindrical polar radius $\rho = \sqrt{x^2 + y^2}$, and cylindrical axis z .

I will eventually be converting this diadic vector basis expansion into a cubic $3 \times 3 \times 3 = 27$ component basis for a **Platonic quaternion** (a standard Rubik's cubes as well as the derivatives) although this is not done in this document; I will work the details to this out at a later time.

From known-to-be-good rock solid linear algebra, the four important properties of a basis exponential, that is, a rotation matrix and demonstrated here equivalently the complex exponential, are

- (1) it has a unity eigenvalue and a corresponding unitary eigenvector (the basis direction, i.e., k -hat for the k -hat exponential **expk**, and similarly for the j -hat exponential **expj** and i -hat exponential **expi**);
- (2) it has determinant 1;
- (3) its inverse is equal to its transpose;
- (4) its conjugate is synonymous with both its inverse and transpose.

The third property is physically equivalent to the reversal of a Rubiks cube rotation: an inverse of a move is equal to transposing it back again.

Mohr's Circle

Example: As an example of an application of the $\hat{k}$ -basis exponential **expk**, here the equations of Mohr's Circle are demonstrated. The equations of Mohr's Circle are obtained with a single application of **expk** as a change of base of the inertia tensor (an R^3 matrix) with all principal inertias I_{xx} , I_{yy} , I_{zz} and the xy-product of inertia I_{xy} . The xy-product of inertia is the component of inertia that corresponds to $i = \hat{k} \times$. The inertia matrix is

$$I = \begin{pmatrix} I_{xx} & I_{xy} & 0 \\ I_{xy} & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{pmatrix}$$

The rotated inertia tensor I' is then the rotated change of base (rotated coordinate axis) around the $\hat{k}$ -axis. This is

$$\begin{aligned} I' &= \begin{pmatrix} I'_{xx} & I'_{xy} & 0 \\ I'_{xy} & I'_{yy} & 0 \\ 0 & 0 & I'_{zz} \end{pmatrix} \\ &= \mathbf{expk}\theta \begin{pmatrix} I_{xx} & I_{xy} & 0 \\ I_{xy} & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{pmatrix} \mathbf{expk}(-\theta) \\ &= \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} I_{xx} & I_{xy} & 0 \\ I_{xy} & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2}(I_{xx} + I_{yy}) + \frac{1}{2}(I_{xx} - I_{yy})\cos 2\theta - I_{xy}\sin 2\theta & \frac{1}{2}(I_{xx} - I_{yy})\sin 2\theta + I_{xy}\cos 2\theta & 0 \\ \frac{1}{2}(I_{xx} - I_{yy})\sin 2\theta + I_{xy}\cos 2\theta & \frac{1}{2}(I_{xx} + I_{yy}) - \frac{1}{2}(I_{xx} - I_{yy})\cos 2\theta + I_{xy}\sin 2\theta & 0 \\ 0 & 0 & I_{zz} \end{pmatrix} \end{aligned}$$

Hence, the rotated principal components are the diagonal entries

$$\begin{aligned} I'_{xx} &= \frac{1}{2}(I_{xx} + I_{yy}) + \frac{1}{2}(I_{xx} - I_{yy})\cos 2\theta - I_{xy}\sin 2\theta \\ I'_{yy} &= \frac{1}{2}(I_{xx} + I_{yy}) - \frac{1}{2}(I_{xx} - I_{yy})\cos 2\theta + I_{xy}\sin 2\theta \\ I'_{zz} &= I_{zz} \end{aligned}$$

and the product of inertia is

$$I'_{xy} = \frac{1}{2}(I_{xx} - I_{yy})\sin 2\theta + I_{xy}\cos 2\theta$$

which agree with the standard formulae from engineering mechanics of solids (Meriam, J.L, and Kraige, L.G. (1986) Engineering Mechanics Vol.1 STATICS 2nd Ed, John Wiley and Sons, NY). It all works like a dream all based upon the deduction that $i = \hat{k} \times$.

This then makes cascades of these transforms (both scleronomic and rheonomic) easy and intuitive to implement, on each of the principal rotation axes of R^3 with the set of congruent similarity changes of base with **expj** and **expi** in addition to **expk**.

Coordinate Systems - Spherical

From the mathematics of the Riemann sphere derived from the j -hat exponential **expj** and k -hat exponential **expk**, the resulting components in each of the basis directions where

$$\begin{aligned} \left(e^{\theta \hat{j} \times}\right)^m \left(e^{\theta \hat{k} \times}\right)^n \hat{i} &= \hat{i} (\cos m\theta)(\cos n\theta) + \hat{j} (\sin n\theta) - \hat{k} (\sin m\theta)(\cos n\theta) \\ \left(e^{\theta \hat{j} \times}\right)^m \left(e^{\theta \hat{k} \times}\right)^n \hat{j} &= -\hat{i} (\cos m\theta)(\sin n\theta) + \hat{j} (\cos n\theta) + \hat{k} (\sin m\theta)(\sin n\theta) \\ \left(e^{\theta \hat{j} \times}\right)^m \left(e^{\theta \hat{k} \times}\right)^n \hat{k} &= \hat{i} (\sin m\theta) + \hat{k} (\cos m\theta) \end{aligned}$$

Weighting the basis by a position vector and expanding diadically the vector basis into a matrix basis for the Riemann sphere results in a spherical coordinate system.

$$\begin{aligned} \left(e^{\theta \hat{j} \times}\right)^m \left(e^{\theta \hat{k} \times}\right)^n \vec{r} &= \left(\hat{i} (\cos m\theta)(\cos n\theta) + \hat{j} (\sin n\theta) - \hat{k} (\sin m\theta)(\cos n\theta)\right) x \\ &\quad + \left(-\hat{i} (\cos m\theta)(\sin n\theta) + \hat{j} (\cos n\theta) + \hat{k} (\sin m\theta)(\sin n\theta)\right) y \\ &\quad + \left(\hat{i} (\sin m\theta) + \hat{j} (0) + \hat{k} (\cos m\theta)\right) z \\ &= \left(\hat{i} (\cos m\theta)(\cos n\theta) + \hat{j} (\sin n\theta) - \hat{k} (\sin m\theta)(\cos n\theta)\right) (\hat{i}' x) \\ &\quad + \left(-\hat{i} (\cos m\theta)(\sin n\theta) + \hat{j} (\cos n\theta) + \hat{k} (\sin m\theta)(\sin n\theta)\right) (\hat{j}' y) \\ &\quad + \left(\hat{i} (\sin m\theta) + \hat{j} (0) + \hat{k} (\cos m\theta)\right) (\hat{k}' z) \\ &= \left(\hat{i} \hat{i}' (\cos m\theta)(\cos n\theta) + \hat{j} \hat{i}' (\sin n\theta) - \hat{k} \hat{i}' (\sin m\theta)(\cos n\theta)\right) (\hat{i} x) \\ &\quad + \left(-\hat{i} \hat{j}' (\cos m\theta)(\sin n\theta) + \hat{j} \hat{j}' (\cos n\theta) + \hat{k} \hat{j}' (\sin m\theta)(\sin n\theta)\right) (\hat{j} y) \\ &\quad + \left(\hat{i} \hat{k}' (\sin m\theta) + \hat{j} \hat{k}' (0) + \hat{k} \hat{k}' (\cos m\theta)\right) (\hat{k} z) \end{aligned}$$

and factoring into matrix times coordinate vector form

$$\begin{aligned} \left(e^{\theta \hat{j} \times}\right)^m \left(e^{\theta \hat{k} \times}\right)^n \vec{r} &= \begin{pmatrix} \left(\hat{i} (\cos m\theta)(\cos n\theta) + \hat{j} (\sin n\theta) - \hat{k} (\sin m\theta)(\cos n\theta)\right) \\ \left(-\hat{i} (\cos m\theta)(\sin n\theta) + \hat{j} (\cos n\theta) + \hat{k} (\sin m\theta)(\sin n\theta)\right) \\ \left(\hat{i} (\sin m\theta) + \hat{j} (0) + \hat{k} (\cos m\theta)\right) \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ &= \begin{pmatrix} (\cos m\theta)(\cos n\theta) \\ (\sin n\theta) \\ -(\sin m\theta)(\cos n\theta) \end{pmatrix} \begin{pmatrix} -(\cos m\theta)(\sin n\theta) \\ (\cos n\theta) \\ (\sin m\theta)(\sin n\theta) \end{pmatrix} \begin{pmatrix} (\sin m\theta) \\ 0 \\ (\cos m\theta) \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ &= \begin{pmatrix} (\cos m\theta)(\cos n\theta) & -(\cos m\theta)(\sin n\theta) & (\sin m\theta) \\ (\sin n\theta) & (\cos n\theta) & 0 \\ -(\sin m\theta)(\cos n\theta) & (\sin m\theta)(\sin n\theta) & (\cos m\theta) \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ &= R_y(m\theta) R_z(n\theta) \vec{r} \end{aligned}$$

that is,

$$e^{\phi \hat{j} \times} e^{\theta \hat{k} \times} (\cdot) = \begin{pmatrix} \cos \phi \cos \theta & -\cos \phi \sin \theta & \sin \phi \\ \sin \theta & \cos \theta & 0 \\ -\sin \phi \cos \theta & \sin \phi \sin \theta & \cos \phi \end{pmatrix} (\cdot)$$

where (1) $\phi = m\theta$ for the j -hat exponential and (2) θ for the k -hat exponential **expk** has been descaled from $n\theta$. This result is the same as multiplying the j -hat exponential **expj** rotation matrix by the k -hat exponential rotation matrix **expk**, that is rotating azimuthal on the inner basis (k -hat) exponential **expk** and then rotating colatitude on the outer basis (j -hat) exponential **expj**. This is intuitive and makes difficult mathematics simple (the great utility of linear algebra). Thus, by matrix multiplication of the j -hat matrix $e^{\phi \hat{j} \times} (\cdot)$ and the k -hat matrix $e^{\theta \hat{k} \times} (\cdot)$ and expanding it all out

$$e^{\phi \hat{j} \times} e^{\theta \hat{k} \times} (\cdot) = \begin{pmatrix} \cos \phi & 0 & \sin \phi \\ 0 & 1 & 0 \\ -\sin \phi & 0 & \cos \phi \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} (\cdot) = \begin{pmatrix} \cos \phi \cos \theta & -\cos \phi \sin \theta & \sin \phi \\ \sin \theta & \cos \theta & 0 \\ -\sin \phi \cos \theta & \sin \phi \sin \theta & \cos \phi \end{pmatrix} (\cdot)$$

So a k -hat exponential **expk** multiplied by another k -hat exponential **expk** is another k -hat exponential **expk** by de-Moivres theorem, and similarly for the j -hat exponential **expj** (with another j -hat exponential **expj**) and i -hat exponential **expi** (with another i -hat exponential **expi**). When the k -hat exponential **expk** is multiplied by a j -hat exponential **expj** or an i -hat exponential **expi**, a spherical system results. Hence, depending on how you choose to parameterise the system, this is a spherical coordinate system with azimuthal angle θ , colatitude angle ϕ , and spherical radius $R = \sqrt{x^2 + y^2 + z^2}$ where the spherical radius R is determined by the x , y , and z coordinates within the position coordinate vector on which **expj** ϕ **expk** θ operates.

Hence this system is also the Riemann sphere and corresponds one-for-one with deriving this sphere through the basis curls [single component cross products, formally known as complex numbers]. Since the basis exponentials are not commutative, the order of multiplication matters and hence results in a differently parameterised spherical coordinate system. A k -hat exponential **expk** multiplied by a j -hat exponential **expj** is different from a j -hat exponential **expj** multiplied by a k -hat exponential **expk**. Since there are three rotation matrices corresponding, respectively, to the k -hat, j -hat, and i -hat exponentials (**expk**, **expj**, and **expi**), and these do not commute, there are ${}^3P_2 = 3 \times 2 = 6$ different spherical matrices that can result. These are

$$\begin{aligned} &e^{\phi \hat{i} \times} e^{\theta \hat{j} \times} (\cdot) \\ &e^{\phi \hat{i} \times} e^{\theta \hat{k} \times} (\cdot) \\ &e^{\phi \hat{j} \times} e^{\theta \hat{i} \times} (\cdot) \\ &e^{\phi \hat{j} \times} e^{\theta \hat{k} \times} (\cdot) \\ &e^{\phi \hat{k} \times} e^{\theta \hat{i} \times} (\cdot) \\ &e^{\phi \hat{k} \times} e^{\theta \hat{j} \times} (\cdot) \end{aligned}$$

As well as their conjugates, single and pairwise which amounts to changing the sign of the arguments or equivalently changing the direction of the rotation (on one or both of azimuth and colatitude). The diadic expansion was for $e^{\phi \hat{j} \times} e^{\theta \hat{k} \times} (\cdot)$ the third last of these. The usual spherical system is that whereby the j -hat exponential **expj** is the inner basis exponential and the k -hat exponential **expk** the outer $e^{\phi \hat{k} \times} e^{\theta \hat{j} \times} (\cdot)$, the last of those listed. The diadic expansion for this is

$$e^{\phi \hat{k} \times} e^{\theta \hat{j} \times} (\cdot) = \begin{pmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{pmatrix} (\cdot) = \begin{pmatrix} \cos \theta \cos \phi & -\sin \phi & \sin \theta \cos \phi \\ \cos \theta \sin \phi & \cos \phi & \sin \theta \sin \phi \\ -\sin \theta & 0 & \cos \theta \end{pmatrix} (\cdot)$$

which would very likely be familiar to the reader. Since the component basis-hat exponentials are rotations, they obey the law of transposes which is synonymous with complex conjugate for $i = k \times, j \times$, and $i \times$. So for example

$$(e^{\phi \hat{k} \times} e^{\theta \hat{j} \times})' = (e^{\theta \hat{j} \times})' (e^{\phi \hat{k} \times})' = e^{-\theta \hat{j} \times} e^{-\phi \hat{k} \times} = \begin{pmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{pmatrix} \begin{pmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix} (\cdot) = \begin{pmatrix} \cos \theta \cos \phi & \cos \theta \sin \phi & -\sin \theta \\ -\sin \phi & \cos \phi & 0 \\ \sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \end{pmatrix} (\cdot)$$

whereby the inner basis-hat rotation becomes the outer basis-hat rotation and the outer basis-hat becomes the inner, and each becomes conjugate. This is a Rubiks cube algorithm in reverse (doing the algorithm backwards). Note that you can try it on a Rubiks cube and see for yourself for $\theta = \phi = \pi/2$. Note that how I have defined the basis vectors, the i -hat, j -hat, and k -hat are column vectors (and the primed i -hat, j -hat, and k -hat are row vectors). This means that the components of each k -hat exponential **expk** in a given basis direction are the column vectors of the k -hat rotation matrix. Similarly for the j -hat exponential, and their multiplication into the spherical coordinates. That is, the components of the multiplication of the k -hat and j -hat exponentials are the columns of the spherical coordinate matrix which is a rotation around the k -hat direction and then a rotation around the j -hat direction. Here is the full working which can also be obtained by matrix multiplying the two rotation matrices in that order (which is how I checked my working from the complex [basis rotation] algebra).

In the i -hat direction

$$\begin{aligned} (e^{\theta \hat{k} \times})^n (e^{\theta \hat{j} \times})^m \hat{i} &= \exp(\theta \hat{k} \times (\cdot))^n \exp(\theta \hat{j} \times (\cdot))^m \hat{i} \\ &= \exp(n\theta \hat{k} \times (\cdot)) \exp(m\theta \hat{j} \times (\cdot)) \hat{i} \\ &= (\cos n\theta + (\sin n\theta) \hat{k} \times) (\cos m\theta + (\sin m\theta) \hat{j} \times) \hat{i} \\ &= (\cos n\theta + (\sin n\theta) \hat{k} \times) ((\cos m\theta) \hat{i} + (\sin m\theta) \hat{j} \times \hat{i}) \\ &= (\cos n\theta + (\sin n\theta) \hat{k} \times) (\hat{i} (\cos m\theta) - \hat{k} (\sin m\theta)) \\ &= (\cos n\theta) (\hat{i} (\cos m\theta) - \hat{k} (\sin m\theta)) + (\sin n\theta) \hat{k} \times (\hat{i} (\cos m\theta) - \hat{k} (\sin m\theta)) \\ &= ((\cos n\theta) \hat{i} (\cos m\theta) - (\cos n\theta) \hat{k} (\sin m\theta)) + ((\sin n\theta) \hat{k} \times \hat{i} (\cos m\theta) - (\sin n\theta) \hat{k} \times \hat{k} (\sin m\theta)) \\ &= \hat{i} (\cos n\theta) (\cos m\theta) + \hat{j} (\sin n\theta) (\cos m\theta) - \hat{k} (\sin m\theta) \end{aligned}$$

In the j -hat direction

$$\begin{aligned}
\left(e^{\theta \hat{k} \times}\right)^n \left(e^{\theta \hat{j} \times}\right)^m \hat{j} &= \exp\left(\theta \hat{k} \times (\cdot)\right)^n \exp\left(\theta \hat{j} \times (\cdot)\right)^m \hat{j} \\
&= \exp\left(n\theta \hat{k} \times (\cdot)\right) \exp\left(m\theta \hat{j} \times (\cdot)\right) \hat{j} \\
&= \left(\text{cosk } n\theta + (\text{sink } n\theta) \hat{k} \times\right) \left(\text{cosj } m\theta + (\text{sinj } m\theta) \hat{j} \times\right) \hat{j} \\
&= \left(\text{cosk } n\theta + (\text{sink } n\theta) \hat{k} \times\right) \left((\text{cosj } m\theta) \hat{j} + (\text{sinj } m\theta) \hat{j} \times \hat{j}\right) \\
&= \left(\text{cosk } n\theta + (\text{sink } n\theta) \hat{k} \times\right) \hat{j} \\
&= \left((\text{cosk } n\theta) \hat{j} + (\text{sink } n\theta) \hat{k} \times \hat{j}\right) \\
&= -\hat{i} (\sin n\theta) + \hat{j} (\cos n\theta)
\end{aligned}$$

In the $\hat{k}$ -hat direction

$$\begin{aligned}
\left(e^{\theta \hat{k} \times}\right)^n \left(e^{\theta \hat{j} \times}\right)^m \hat{k} &= \exp\left(n\theta \hat{k} \times (\cdot)\right) \exp\left(m\theta \hat{j} \times (\cdot)\right) \hat{k} \\
&= \left(\text{cosk } n\theta + (\text{sink } n\theta) \hat{k} \times\right) \left(\text{cosj } m\theta + (\text{sinj } m\theta) \hat{j} \times\right) \hat{k} \\
&= \left(\text{cosk } n\theta + (\text{sink } n\theta) \hat{k} \times\right) \left((\text{cosj } m\theta) \hat{k} + (\text{sinj } m\theta) \hat{j} \times \hat{k}\right) \\
&= \left(\text{cosk } n\theta + (\text{sink } n\theta) \hat{k} \times\right) \left(\hat{k} (\cos m\theta) + \hat{i} (\sin m\theta)\right) \\
&= \left((\text{cosk } n\theta) \hat{k} (\cos m\theta) + (\text{cosk } n\theta) \hat{i} (\sin m\theta)\right) + \left((\text{sink } n\theta) \hat{k} \times \hat{k} (\cos m\theta) + (\text{sink } n\theta) \hat{k} \times \hat{i} (\sin m\theta)\right) \\
&= \hat{i} (\cos n\theta) (\sin m\theta) + \hat{j} (\sin n\theta) (\sin m\theta) + \hat{k} (\cos m\theta)
\end{aligned}$$

The components are therefore

$$\begin{aligned}
\left(e^{\theta \hat{k} \times}\right)^n \left(e^{\theta \hat{j} \times}\right)^m \hat{i} &= \hat{i} (\cos n\theta) (\cos m\theta) + \hat{j} (\sin n\theta) (\cos m\theta) - \hat{k} (\sin m\theta) \\
\left(e^{\theta \hat{k} \times}\right)^n \left(e^{\theta \hat{j} \times}\right)^m \hat{j} &= -\hat{i} (\sin n\theta) + \hat{j} (\cos n\theta) \\
\left(e^{\theta \hat{k} \times}\right)^n \left(e^{\theta \hat{j} \times}\right)^m \hat{k} &= \hat{i} (\cos n\theta) (\sin m\theta) + \hat{j} (\sin n\theta) (\sin m\theta) + \hat{k} (\cos m\theta)
\end{aligned}$$

where $\left(e^{\theta \hat{k} \times}\right)^n \left(e^{\theta \hat{j} \times}\right)^m = e^{n\theta \hat{k} \times} e^{m\theta \hat{j} \times}$. With standard azimuth and colatitude arguments ϕ and θ replacing $n\theta$ and $m\theta$ these are

$$\begin{aligned}
e^{\phi \hat{k} \times} e^{\theta \hat{j} \times} \hat{i} &= \hat{i} \cos \phi \cos \theta + \hat{j} \sin \phi \cos \theta - \hat{k} \sin \theta \\
e^{\phi \hat{k} \times} e^{\theta \hat{j} \times} \hat{j} &= -\hat{i} \sin \phi + \hat{j} \cos \phi \\
e^{\phi \hat{k} \times} e^{\theta \hat{j} \times} \hat{k} &= \hat{i} \cos \phi \sin \theta + \hat{j} \sin \phi \sin \theta + \hat{k} \cos \theta
\end{aligned}$$

with a diadic expansion of a position vector basis into a matrix basis, which essentially places each of the three vectors down the columns as the scalar components of the matrix basis

$$\begin{aligned}
e^{\phi \hat{k} \times} e^{\theta \hat{j} \times} \vec{r} &= (\hat{i} \cos \phi \cos \theta + \hat{j} \sin \phi \cos \theta - \hat{k} \sin \theta) x + (-\hat{i} \sin \phi + \hat{j} \cos \phi + \hat{k} 0) y + (\hat{i} \cos \phi \sin \theta + \hat{j} \sin \phi \sin \theta + \hat{k} \cos \theta) z \\
&= (\hat{i} \cos \phi \cos \theta + \hat{j} \sin \phi \cos \theta - \hat{k} \sin \theta) (\hat{i}' x) \\
&\quad + (-\hat{i} \sin \phi + \hat{j} \cos \phi + \hat{k} 0) (\hat{j}' y) + (\hat{i} \cos \phi \sin \theta + \hat{j} \sin \phi \sin \theta + \hat{k} \cos \theta) (\hat{k}' z) \\
&= (\hat{i} \hat{i}' \cos \phi \cos \theta + \hat{j} \hat{i}' \sin \phi \cos \theta - \hat{k} \hat{i}' \sin \theta) (\hat{i} x) \\
&\quad + (-\hat{i} \hat{j}' \sin \phi + \hat{j} \hat{j}' \cos \phi + \hat{k} \hat{j}' 0) (\hat{j} y) + (\hat{i} \hat{k}' \cos \phi \sin \theta + \hat{j} \hat{k}' \sin \phi \sin \theta + \hat{k} \hat{k}' \cos \theta) (\hat{k} z) \\
&= \left((\hat{i} \cos \phi \cos \theta + \hat{j} \sin \phi \cos \theta - \hat{k} \sin \theta), (-\hat{i} \sin \phi + \hat{j} \cos \phi + \hat{k} 0), (\hat{i} \cos \phi \sin \theta + \hat{j} \sin \phi \sin \theta + \hat{k} \cos \theta) \right) \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\
&= \begin{pmatrix} \cos \phi \cos \theta & -\sin \phi \cos \phi \sin \theta \\ \sin \phi \cos \theta & \cos \phi & \sin \phi \sin \theta \\ -\sin \theta & 0 & \cos \theta \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}
\end{aligned}$$

that is,

$$e^{\phi \hat{k} \times} e^{\theta \hat{j} \times} (\cdot) = \begin{pmatrix} \cos \phi \cos \theta & -\sin \phi \cos \phi \sin \theta \\ \sin \phi \cos \theta & \cos \phi & \sin \phi \sin \theta \\ -\sin \theta & 0 & \cos \theta \end{pmatrix} (\cdot)$$

which is the standard spherical coordinate system as earlier by multiplying the k -hat exponential **expk** ϕ by the j -hat exponential **expj** θ and considering the resulting product as the spherical operator for a position coordinate vector in the azimuth angle ϕ and colatitude angle θ (or however one wants to parameterize it; I actually quite like θ for azimuth and ϕ for colatitude).

Spin, Precession, and Nutation

These are rotations on the three mutually perpendicular axes. Since there are 3 basis vectors for 3 axes, there are $3! = 3 \times 2 \times 1 = 6$ such matrices after diadic vector expansion. Since the basis-hat exponentials don't commute, the six ($3! = 6$) systems are

$$\begin{aligned}
&e^{\alpha \hat{i} \times} e^{\beta \hat{j} \times} e^{\gamma \hat{k} \times} (\cdot) \\
&e^{\alpha \hat{i} \times} e^{\gamma \hat{k} \times} e^{\beta \hat{j} \times} (\cdot) \\
&e^{\beta \hat{j} \times} e^{\gamma \hat{k} \times} e^{\alpha \hat{i} \times} (\cdot) \\
&e^{\beta \hat{j} \times} e^{\alpha \hat{i} \times} e^{\gamma \hat{k} \times} (\cdot) \\
&e^{\gamma \hat{k} \times} e^{\alpha \hat{i} \times} e^{\beta \hat{j} \times} (\cdot) \\
&e^{\gamma \hat{k} \times} e^{\beta \hat{j} \times} e^{\alpha \hat{i} \times} (\cdot)
\end{aligned}$$

assuming that each of the axes are free to rotate independently, that is, with three rotational degrees of freedom. Taking the last of the six to be the standard spin, precession, and nutation system

$$e^{\gamma \hat{k} \times} e^{\beta \hat{j} \times} e^{\alpha \hat{i} \times} (\cdot) = \begin{pmatrix} \cos \gamma & -\sin \gamma & 0 \\ \sin \gamma & \cos \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \beta & 0 & \sin \beta \\ 0 & 1 & 0 \\ -\sin \beta & 0 & \cos \beta \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & -\sin \alpha \\ 0 & \sin \alpha & \cos \alpha \end{pmatrix} (\cdot)$$

$$= \begin{pmatrix} \cos \beta \cos \gamma & \sin \alpha \sin \beta \cos \gamma - \cos \alpha \sin \gamma & \cos \alpha \sin \beta \cos \gamma + \sin \alpha \sin \gamma \\ \cos \beta \sin \gamma & \cos \alpha \cos \gamma + \sin \alpha \sin \beta \sin \gamma & \cos \alpha \sin \beta \sin \gamma - \sin \alpha \cos \gamma \\ -\sin \beta & \sin \alpha \cos \beta & \cos \alpha \cos \beta \end{pmatrix} (\cdot)$$

by matrix multiplication of the rotation matrices on my TI workhorse. I also calculated these by expanding out the basis-hat exponentials with k -cross, j -cross, and i -cross, equivalent to complex algebra, for each basis direction and diadically expanding but this is very time consuming and subject to much mathematical frustration (but doable nevertheless and results in the same system given that the complex algebra method with is now rock solid and corresponds with R^3 linear algebra one-for-one). Time will tell how useful the additional nutation rotation is in the context of Euler inertia equations for example.

Long Chain Diffusion Equations

These multiplications can be continued to higher order since only if a multiplication is by its own basis-hat or conjugate does it simplify. This means that long chains of the basis-hat exponentials can form as continued products of the rotations on the mutually perpendicular axes of R^3 . These form long chain groups and are also differential equations. God's number on a Rubiks cube is 20, so that is a good ballpark guess to how large these chains would typically get for a complete enumeration of a Rubiks cube when all rotations are perfect quarter turns (quartic roots of unity). Any component basis-hat rotation in one of these chains can be scaled for a frequency allowing potentially for infinite sophistication.

Quarter turn rotations

This sub-topic will show that $\hat{k} \times$ and $e^{\frac{\pi}{2} \hat{k} \times}$ (the k -hat exponential at $\pi/2$) are not equivalent [whenever k -hat is recognised], despite $e^{\frac{\pi}{2} \hat{i}} = i$ in complex theory, and try to reconcile this difficulty and then extend for further theory. The issue is that $\hat{k} \times$ is singular whilst $e^{\frac{\pi}{2} \hat{k} \times}$ is not since the latter has the k -hat eigenvalue whilst for $\hat{k} \times$ it is null. It is also true that $e^{\pi \hat{k} \times}$ (the k -hat exponential at π) is not equal to -1 ; in the i -hat direction, it equals -1 , in the j -hat direction it equals -1 , and in the k -hat direction it equals $+1$ so by dyadic expansion

$$e^{\pi \hat{k} \times} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{i.e., not equal to} \quad -1I = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

For the quartic roots of unity, the quarter (right angle) rotations of Rubiks cubes for example, the k -hat matrix becomes

$$e^{\frac{\pi}{4} \hat{k} \times} (\cdot) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} (\cdot) \quad \text{which can also be expressed as the diads } e^{\frac{\pi}{4} \hat{k} \times} (\cdot) = (-\hat{i} \hat{j}' + \hat{j} \hat{i}' + \hat{k} \hat{k}') (\cdot)$$

If the operator $\hat{k} \times$ is diadically expanded it is

$$\hat{k} \times (\cdot) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} (\cdot)$$

that is, in the i -hat direction, j -hat direction, and k -hat direction

$$(\hat{k} \times) \hat{i} = \hat{i}(0) + \hat{j}(1) + \hat{k}(0)$$

$$(\hat{k} \times) \hat{j} = \hat{i}(-1) + \hat{j}(0) + \hat{k}(0)$$

$$(\hat{k} \times) \hat{k} = \hat{i}(0) + \hat{j}(0) + \hat{k}(0)$$

which form the columns of the diadically expanded vectors

$$\begin{aligned} \hat{k} \times \vec{x} &= (\hat{i}(0) + \hat{j}(1) + \hat{k}(0))x + (\hat{i}(-1) + \hat{j}(0) + \hat{k}(0))y + (\hat{i}(0) + \hat{j}(0) + \hat{k}(0))z \\ &= (\hat{i}(0) + \hat{j}(1) + \hat{k}(0))(\hat{i}' \hat{i})x + (\hat{i}(-1) + \hat{j}(0) + \hat{k}(0))(\hat{j}' \hat{j})y + (\hat{i}(0) + \hat{j}(0) + \hat{k}(0))(\hat{k}' \hat{k})z \\ &= (\hat{i} \hat{i}'(0) + \hat{j} \hat{i}'(1) + \hat{k} \hat{i}'(0))(\hat{i} x) + (\hat{i} \hat{j}'(-1) + \hat{j} \hat{j}'(0) + \hat{k} \hat{j}'(0))(\hat{j} y) + (\hat{i} \hat{k}'(0) + \hat{j} \hat{k}'(0) + \hat{k} \hat{k}'(0))(\hat{k} z) \\ &= \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \end{aligned}$$

So then the question that maybe asked is where did the eigenvalue (otherwise in the $\hat{k} \hat{k}'$ position) go? This is the unitary value (for

$\hat{k} \hat{k}'$) provided by the **cosk** of the k -hat cosine for its own basis direction. This fact means that $\hat{k} \times (\cdot)$ and $e^{\frac{\pi}{2} \hat{k} \times} (\cdot)$ are not equivalent,

despite what Euler's formula for a quarter rotation ($e^{\frac{\pi}{2} i} = i$) might suggest, whenever the z component is not zero [or ignored as latent]. This is because the **cosk** part is independent of argument in the k -hat direction (including at $\theta = \pi/2$). The k -cross operator is not invertible since it is singular, that is, it does not have an inverse; although it does have a transpose and this is what the inverse would be if it existed which is essentially just a sign change similarly as cubing $\sqrt{-1}$, i.e., $(\sqrt{-1})^3 = -\sqrt{-1}$. But the complex number is inverted all the time. For these equations I think this is because either $\hat{k} \times (\cdot)$ is being moved to the other side of an equation and manipulated, and then implicitly moved back again with powers, i.e., multiplying both sides by $(\hat{k} \times)^{-3} = -\hat{k} \times$. Either that or the mathematics is implicitly using $e^{\frac{\pi}{2} \hat{k} \times} (\cdot)$. These features are similarly for the j -hat and i -hat exponentials, and the eigenvalues in the $\hat{j} \hat{j}'$ and $\hat{i} \hat{i}'$ positions respectively. For the quartic roots of unity on their respective axes, these are

$$e^{\frac{\pi}{2} \hat{j} \times} (\cdot) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} (\cdot) \quad \text{and} \quad e^{\frac{\pi}{2} \hat{i} \times} (\cdot) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} (\cdot)$$

whilst j -cross and i -cross are the same except without the eigenvalues in their own basis direction

$$\hat{j} \times (\cdot) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} (\cdot) \quad \text{and} \quad \hat{i} \times (\cdot) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} (\cdot)$$

Each of basis-hat quarter rotations, which include the rotational eigenvalue, have three unitary entries and six zeroes. The unitary eigenvalue, a one corresponding to the negative **sink** and a one corresponding to the positive **sink**. Any product of any of these three rotation matrices $e^{\frac{\pi}{2} \hat{k} \times}(\cdot)$, $e^{\frac{\pi}{2} \hat{j} \times}(\cdot)$, or $e^{\frac{\pi}{2} \hat{i} \times}(\cdot)$ always retain 3 ones, that is, there always remains three unitary values (two +1s and one -1) and six zeroes (for the 9 scalar components associated with the 9 matrix basis components). This means that there are considerable simplifications for long non-commutative chains. These are the proper full rank R^3 spin matrices. For example the k -hat quartic rotation times the j -hat quartic rotation times the i -hat rotation, that for spin-precession-nutation but for discrete quartic rotations, then becomes

$$e^{\frac{\pi}{2} \hat{k} \times} e^{\frac{\pi}{2} \hat{j} \times} e^{\frac{\pi}{2} \hat{i} \times} (\cdot) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} (\cdot) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} (\cdot)$$

The square is then

$$\left(e^{\frac{\pi}{2} \hat{k} \times} e^{\frac{\pi}{2} \hat{j} \times} e^{\frac{\pi}{2} \hat{i} \times} \right)^2 (\cdot) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} (\cdot) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} (\cdot)$$

and cubic power

$$\left(e^{\frac{\pi}{2} \hat{k} \times} e^{\frac{\pi}{2} \hat{j} \times} e^{\frac{\pi}{2} \hat{i} \times} \right)^3 (\cdot) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} (\cdot) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} (\cdot) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} (\cdot)$$

and finally the quartic power returns the identity (similarly as the isometric quartic roots of unity for each of the R^3 axes $i = \hat{k} \times, \hat{j} \times, \hat{i} \times$)

$$\left(e^{\frac{\pi}{2} \hat{k} \times} e^{\frac{\pi}{2} \hat{j} \times} e^{\frac{\pi}{2} \hat{i} \times} \right)^4 (\cdot) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} (\cdot) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} (\cdot) = I(\cdot)$$

so this triple rotation product forms a closed operator and group of order 4 (a quartic root of unity that visits each of the three R^3 axes). There are many of these of all orders of products. If the so-called God's number is anything to go by, which is the maximum number of turns to return a scrambled cube to the solved state, the order of such operators extend to 20 such multiplications to form a closed group. This operator would then be comprised an order twenty sub-group, and taken as a single operator (as the examples above), a group of order two, that is, an algorithm to scrambled state and algorithm back to solved state that contains null redundancies (no repeats of subgroups, which is important for efficiency of solution algorithms). The simplest such order two groups are the double crosses, k -hat double cross ($k \times k \times$), j -hat double cross ($j \times j \times$), and i -hat double cross ($i \times i \times$) whilst the algorithm for God's number

would be the most sophisticated (once it is found). Ideally, I would like every single one of the enumerated (although, for the mathematics, this needs the further refinement of being able to select a layer... I have worked this out already but have to type it out some time in the future).

Rubiks Cube (Platonic Quaternion) Bit Position Tracking

The $\pi/2$ k -hat, $\pi/2$ j -hat, and $\pi/2$ i -hat exponentials for quarter rotations track bit positions of a Rubiks cube exactly, that is, for any **face/center**, **edge**, or **corner** bit. This is either by left to right multiplication to a row position vector or right to left multiplication to a column position vector. These do not keep track of orientation (rather than position) which needs a further equation (do now, with what I have defined as **colour vectors** later in this document), and cannot as of this time track sets of bits to fully implement Rubiks algorithms mathematically.

Note that I was originally using k -cross ($\mathbf{k} \times$), j -cross ($\mathbf{j} \times$), and i -cross ($\mathbf{i} \times$) for bit position tracking but then after each move I would have to add back the k -hat component for a k -cross move (and similarly the j -hat component for a j -cross move and the i -hat component for an i -cross move) which I now know is the rotational eigenvalue (which is nulled by the basis-crosses as these are singular). This problem I was having has hence been solved by the k -hat, j -hat, and i -hat exponentials (which are full rank) which retain the rotational eigenvalue importantly here for $\pi/2$ rotations on each of the R^3 three axes. This demonstrates that complex i and the complex exponential at $\pi/2$ are not equivalent in the complete theory of basis rotational operators (I define these in a later document as the singular complex numbers and the full rank complex numbers respectively based on this document).

Similarly, just for interest, $e^{\pi \hat{k} \times}$ (the k -hat exponential at π) is not equal to -1 when considered as an operator and k -hat is not latent, it is rather equal to

$$e^{\pi \hat{k} \times} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which is the basis equivalent to $e^{\pi i} = -1$ with $i = \hat{k} \times$. Letting each **face/center** bit be one unit from the origin. The **edges** are then $\sqrt{2}$ units from the origin (planar Pythagoras) and the **corners** are $\sqrt{3}$ units from the origin (volume Pythagoras). The bit position vectors:

The position vectors of the (3axes $\times$ 2signs =) 6 **faces/center** bits are

+x [1,0,0], +y [0,1,0], +z [0,0,1], -x [-1,0,0], -y [0,-1,0], and -z [0,0,-1]

The position vectors, ignoring the normalization distance factor of $\sqrt{2}$, of the 12 **edge** bits are

top slice:	+1 offset x-y plane (at z=+1)	[1,0,+1], [-1,0,+1], [0,1,+1], [0,-1,+1]	(k -hat spin)
middle slice:	the x-y plane (at z=0)	[1,1, 0], [1,-1, 0], [-1,1,0], [-1,-1,0]	(k -hat spin)
bottom slice:	-1 offset x-y plane (at z=-1)	[1,0,-1], [-1,0,-1], [0,1,-1], [0,-1,-1]	(k -hat spin)

The position vectors, ignoring the normalization distance factor of $\sqrt{3}$, of the (2signs $\times$ 2signs $\times$ 2signs =) 8 **corner** bits are

top 4 corners (z=+1) [1,1,+1], [1,-1,+1], [-1,1,+1], [-1,-1,+1]
 bottom 4 corners (z=-1) [1,1,-1], [1,-1,-1], [-1,1,-1], [-1,-1,-1]

Here are three examples, one for each type of bit, that takes a bit moves it around and then returns it to the starting position:

center bit example: [1,0,0] expk×expj×expj'=[1,0,0] or expk×expj×expj'× [1,0,0]'=[1,0,0]
edge bit example: [1,1,0] expi×expk×expk×expi'×expj×expj=[1,1,0] or expi×expk×expk×expi'×expj×expj× [1,1,0]'=[1,1,0]'
corner bit example: [1,1,1] expk×expj×expk'×expi=[1,1,1] or expk×expj×expk'×expi× [1,1,1]'=[1,1,1]'

center bit example: $(1 \ 0 \ 0) e^{\frac{\pi}{2} \hat{k} \times} e^{\frac{\pi}{2} \hat{j} \times} \left(e^{\frac{\pi}{2} \hat{j} \times} \right)^T = (1 \ 0 \ 0) e^{\frac{\pi}{2} \hat{k} \times} e^{\frac{\pi}{2} \hat{i} \times} e^{-\frac{\pi}{2} \hat{j} \times} = (1 \ 0 \ 0)$

edge bit example: $(1 \ 1 \ 0) e^{\frac{\pi}{2} \hat{j} \times} \left(e^{\frac{\pi}{2} \hat{k} \times} \right)^2 \left(e^{\frac{\pi}{2} \hat{i} \times} \right)^T \left(e^{\frac{\pi}{2} \hat{j} \times} \right)^2 = (1 \ 1 \ 0) e^{\frac{\pi}{2} \hat{i} \times} e^{\pi \hat{k} \times} e^{-\frac{\pi}{2} \hat{i} \times} e^{\pi \hat{j} \times} = (1 \ 1 \ 0)$

corner bit example: $(1 \ 1 \ 1) e^{\frac{\pi}{2} \hat{k} \times} e^{\frac{\pi}{2} \hat{j} \times} \left(e^{\frac{\pi}{2} \hat{k} \times} \right)^T e^{\frac{\pi}{2} \hat{i} \times} = (1 \ 1 \ 1) e^{\frac{\pi}{2} \hat{k} \times} e^{\frac{\pi}{2} \hat{j} \times} e^{-\frac{\pi}{2} \hat{k} \times} e^{\frac{\pi}{2} \hat{i} \times} = (1 \ 1 \ 1)$

or, the same but with column rather than row position vectors

center $e^{\frac{\pi}{2} \hat{k} \times} e^{\frac{\pi}{2} \hat{i} \times} \left(e^{\frac{\pi}{2} \hat{j} \times} \right)^T \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}^T \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

edge $e^{\frac{\pi}{2} \hat{i} \times} \left(e^{\frac{\pi}{2} \hat{k} \times} \right)^2 \left(e^{\frac{\pi}{2} \hat{i} \times} \right)^T \left(e^{\frac{\pi}{2} \hat{j} \times} \right)^2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}^T \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

corner $e^{\frac{\pi}{2} \hat{k} \times} e^{\frac{\pi}{2} \hat{j} \times} \left(e^{\frac{\pi}{2} \hat{k} \times} \right)^T e^{\frac{\pi}{2} \hat{i} \times} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}^T \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

demonstrating that the $\hat{k}$ -hat exponential **expk** (equivalent to the complex exponential in the real R^3 vector volume) really is the Queen of mathematics, and also demonstrating that the basis-hat exponential equations do what they are supposed to, in this case move the moved bit back to its starting position (albeit with differing orientations... now overcome with colour vectors). Note for working proficiency that transposing a basis-hat exponential simply swaps the signs of the two non-diagonal entries (the negative curl becomes positive and the positive curl becomes negative) since the transpose is equivalent to both the conjugate (transpose) and the inverse (which is the negative power by de-Moivre's theorem).

These can be verified physically on a Rubik's cube (and even more easily programmed). Note that if you do want to try the equations out physically on Rubik's cube [which is how I knew these algorithms would return the bits to their starting positions], then you have to know how to solve it (at least by the beginners method) as you may have to repeatedly get it back to the solved state (after making a wrong turn for example). Learning the beginners method takes about a week and is much like learning to touch type (it is not something that comes naturally).

The mathematics of the k -hat exponential (for the bit tracking equations) is a mighty start to a full mathematical elucidation of Rubik's marvellous invention. I think Plato, Aristotle, Archimedes, Euclid, and Pythagoras would have loved Rubik's cubes and derivatives (Platonic quaternions) almost as much as I do.

Further Expressions for sink and cosk, and Defining Basis Tangent and Secant, tank and seck

The diadic expansions for k -cross and the k -hat exponential are

$$\hat{k} \times (\cdot) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} (\cdot) \quad \text{and} \quad e^{\theta \hat{k} \times} (\cdot) = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} (\cdot)$$

the operators **cosk** and **sink** can be similarly diadically expanded into a matrix basis. These are then verified with reference to the above for the expressions for k -cross and the k -hat exponential with Euler's formula, and provide a means to define **tank** (that is, basis tangent for $i = \hat{k} \times$, "tan-kay", rather than an armoured vehicle or water cistern). For **cosk**, in the i -hat, j -hat, and k -hat directions

$$(\mathbf{cosk} \theta) \hat{i} = \hat{i}(\cos \theta) + \hat{j}(0) + \hat{k}(0)$$

$$(\mathbf{cosk} \theta) \hat{j} = \hat{i}(0) + \hat{j}(\cos \theta) + \hat{k}(0)$$

$$(\mathbf{cosk} \theta) \hat{k} = \hat{i}(0) + \hat{j}(0) + \hat{k}(1)$$

which is then in a matrix basis (the vectors being the columns in the matrix basis) for the diadic expansion

$$\mathbf{cosk} \theta (\cdot) = \begin{pmatrix} \cos \theta & 0 & 0 \\ 0 & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} (\cdot)$$

Similarly for the k -hat sine. So for **sink**, in the i -hat, j -hat, and k -hat directions

$$(\mathbf{sink} \theta) \hat{i} = \hat{i}(\sin \theta) + \hat{j}(0) + \hat{k}(0)$$

$$(\mathbf{sink} \theta) \hat{j} = \hat{i}(0) + \hat{j}(\sin \theta) + \hat{k}(0)$$

$$(\mathbf{sink} \theta) \hat{k} = \hat{i}(0) + \hat{j}(0) + \hat{k}(\theta)$$

which is then in a matrix basis for the diadic expansion

$$\mathbf{sink} \theta (\cdot) = \begin{pmatrix} \sin \theta & 0 & 0 \\ 0 & \sin \theta & 0 \\ 0 & 0 & \theta \end{pmatrix} (\cdot)$$

where the component vectors have mapped to the columns in the matrix basis. Note that in the Euler formula, it is the k -hat sine times the k -cross, so

$$(\mathbf{sink}\theta)\hat{k} \times (\cdot) = \begin{pmatrix} \sin \theta & 0 & 0 \\ 0 & \sin \theta & 0 \\ 0 & 0 & \theta \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} (\cdot) = \begin{pmatrix} 0 & -\sin \theta & 0 \\ \sin \theta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} (\cdot)$$

which shows how k -cross works and why I called it a basis curl, that is, it has curled the sines into the curl position on the anti-diagonals (and nulled the eigenvalue so it is not double counted when added to the k -hat cosine). So for Euler's formula, adding the k -hat cosine to the product of the k -hat sine and k -cross

$$\begin{aligned} \mathbf{cosk}\theta + (\mathbf{sink}\theta)\hat{k} \times (\cdot) &= \begin{pmatrix} \cos \theta & 0 & 0 \\ 0 & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} \sin \theta & 0 & 0 \\ 0 & \sin \theta & 0 \\ 0 & 0 & \theta \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} (\cdot) = \begin{pmatrix} \cos \theta & 0 & 0 \\ 0 & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -\sin \theta & 0 \\ \sin \theta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} (\cdot) \\ &= \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} (\cdot) = e^{\theta \hat{k} \times} \end{aligned}$$

which verifies the correctness of the diadic expansions for **cosk** and **sink**. This means they can also be obtained by the addition and subtraction and the k -hat exponential and its conjugate (inverse). For **cosk**

$$\begin{aligned} \frac{1}{2} (e^{\theta \hat{k} \times} + e^{-\theta \hat{k} \times}) &= \frac{1}{2} \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2 \cos \theta & 0 & 0 \\ 0 & 2 \cos \theta & 0 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} \cos \theta & 0 & 0 \\ 0 & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} (\cdot) \\ &= \mathbf{cosk}(\theta) \end{aligned}$$

For **sink** times k -cross

$$\begin{aligned} \frac{1}{2} (e^{\theta \hat{k} \times} - e^{-\theta \hat{k} \times}) &= \frac{1}{2} \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 & -2 \sin \theta & 0 \\ 2 \sin \theta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\sin \theta & 0 \\ \sin \theta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} (\cdot) \\ &= (\mathbf{sink}\theta)\hat{k} \times (\cdot) \end{aligned}$$

so the problem here is that k -cross is not invertible so it does not make sense to define $\mathbf{sink}\theta = \frac{1}{2} (e^{\theta \hat{k} \times} - e^{-\theta \hat{k} \times}) (\hat{k} \times)^{-1} (\cdot)$. It turns out it doesn't really matter (at least I think) since it has been deduced already by diadic expansion and all of the equations give consistent results without having to make an inversion of k -cross. What all this does mean is that it is not right to arbitrarily replace k -cross by the k -hat exponential **expk** at $\theta = \pi/2$ without bearing in mind that there may be a problem down the line (for programming such a function into a calculator for example) where one is singular and the other is full rank.

The k -hat cosine **cosk** is non-singular, since it has the rotation eigenvalue, whilst the k -hat sine **sink** coupled with $k \times$ is singular. The k -hat cosine **cosk** can be inverted so it is possible to define the k -hat tangent, **tank** ("tan-kay"), and the k -hat tangent times k -cross. The inverse of **cosk** is **seck** ("sec-kay"), the k -hat basis secant

$$\mathbf{seck}\theta = \mathbf{cosk}^{-1}\theta(\cdot) = \begin{pmatrix} \cos\theta & 0 & 0 \\ 0 & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1}(\cdot) = \begin{pmatrix} \cos^{-1}\theta & 0 & 0 \\ 0 & \cos^{-1}\theta & 0 \\ 0 & 0 & 1^{-1} \end{pmatrix}(\cdot) = \begin{pmatrix} \sec\theta & 0 & 0 \\ 0 & \sec\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}(\cdot)$$

so the k -hat tangent **tank** is then defined as the product of **seck** and **sink** as

$$\begin{aligned} \mathbf{tank}\theta &= (\mathbf{cosk}\theta)^{-1}(\mathbf{sink}\theta) = \begin{pmatrix} \sec\theta & 0 & 0 \\ 0 & \sec\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sin\theta & 0 & 0 \\ 0 & \sin\theta & 0 \\ 0 & 0 & \theta \end{pmatrix}(\cdot) \\ &= \begin{pmatrix} \tan\theta & 0 & 0 \\ 0 & \tan\theta & 0 \\ 0 & 0 & \theta \end{pmatrix}(\cdot) \end{aligned}$$

Power series expansion of **tank** with a diadic expansion yields the same result. Also,

$$\begin{aligned} (\mathbf{tank}\theta)\hat{k} \times &= (e^{\theta\hat{k} \times} + e^{\theta\hat{k} \times})^{-1}(e^{\theta\hat{k} \times} - e^{\theta\hat{k} \times}) = \begin{pmatrix} \tan\theta & 0 & 0 \\ 0 & \tan\theta & 0 \\ 0 & 0 & \theta \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}(\cdot) \\ &= \begin{pmatrix} 0 & -\tan\theta & 0 \\ \tan\theta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}(\cdot) \end{aligned}$$

which has the same matrix components (in positions $\hat{i}\hat{j}'$ and $\hat{j}\hat{i}'$) as the k -hat sine **sink** (with scalar tangents rather than sines). The congruent basis functions (which are similarity invariants) for j -hat and i -hat are **secj**, **tanj**, **seci**, and **tani**. For the j -hat axis, the basis secant **secj** (“sec-jay”) and basis tangent **tanj** (“tan-jay”) are

$$\mathbf{secj}\theta = \begin{pmatrix} \cos\theta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cos\theta \end{pmatrix}^{-1} = \begin{pmatrix} \sec\theta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sec\theta \end{pmatrix} \quad \text{and} \quad \mathbf{tanj}\theta = \begin{pmatrix} \tan\theta & 0 & 0 \\ 0 & \theta & 0 \\ 0 & 0 & \tan\theta \end{pmatrix}$$

For the i -hat axis, the basis secant **seci** (“sec-eye”) and basis tangent **tani** (“tan-eye”) are

$$\mathbf{seci}\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & 0 \\ 0 & 0 & \cos\theta \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sec\theta & 0 \\ 0 & 0 & \sec\theta \end{pmatrix} \quad \text{and} \quad \mathbf{tanj}\theta = \begin{pmatrix} \theta & 0 & 0 \\ 0 & \tan\theta & 0 \\ 0 & 0 & \tan\theta \end{pmatrix}$$

I have also made the definitions in terms of $i = \hat{k} \times$

$$\mathbf{sink}\theta = \frac{1}{2} \int_{-\theta}^{\theta} e^{y\hat{k} \times} dy \quad \text{for the } k\text{-hat sine (the vertical oscillator)}$$

$$\mathbf{cosk}\theta = \frac{1}{2} \frac{d}{d\theta} \int_{-\theta}^{\theta} e^{y\hat{k} \times} dy \quad \text{for the } k\text{-hat cosine (the horizontal oscillator)}$$

These similarly give the right expressions for the diadic expansions into matrix bases. For the xy -plane, for **sink**

$$\mathbf{sink} \theta = \frac{1}{2} \int_{-\theta}^{\theta} e^{y \hat{k} \times} dy = \frac{1}{2} \int_{-\theta}^{\theta} \begin{pmatrix} \cos y & -\sin y & 0 \\ \sin y & \cos y & 0 \\ 0 & 0 & 1 \end{pmatrix} dy = \begin{pmatrix} \frac{1}{2} \int_{-\theta}^{\theta} \cos y dy & -\frac{1}{2} \int_{-\theta}^{\theta} \sin y dy & 0 \\ \frac{1}{2} \int_{-\theta}^{\theta} \sin y dy & \frac{1}{2} \int_{-\theta}^{\theta} \cos y dy & 0 \\ 0 & 0 & \frac{1}{2} \int_{-\theta}^{\theta} dy \end{pmatrix} = \begin{pmatrix} \sin \theta & 0 & 0 \\ 0 & \sin \theta & 0 \\ 0 & 0 & \theta \end{pmatrix}$$

and for **cosk**

$$\mathbf{cosk} \theta = \frac{1}{2} \frac{d}{d\theta} \int_{-\theta}^{\theta} e^{y \hat{k} \times} dy = \frac{d}{d\theta} \begin{pmatrix} \sin \theta & 0 & 0 \\ 0 & \sin \theta & 0 \\ 0 & 0 & \theta \end{pmatrix} = \begin{pmatrix} \frac{d}{d\theta} \sin \theta & 0 & 0 \\ 0 & \frac{d}{d\theta} \sin \theta & 0 \\ 0 & 0 & \frac{d}{d\theta} \theta \end{pmatrix} = \begin{pmatrix} \cos \theta & 0 & 0 \\ 0 & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

These formulae are therefore alternative ways of defining **sink** and **cosk**. The formula are similar for the standard basis similarity isometries of **sink** and **cosk**, those of **sinj**, **cosj**, **sini**, and **cosi**. For the zx -plane, for **sinj**

$$\mathbf{sinj} \theta = \frac{1}{2} \int_{-\theta}^{\theta} e^{y \hat{j} \times} dy = \frac{1}{2} \int_{-\theta}^{\theta} \begin{pmatrix} \cos y & 0 & \sin y \\ 0 & 1 & 0 \\ -\sin y & 0 & \cos y \end{pmatrix} dy = \begin{pmatrix} \frac{1}{2} \int_{-\theta}^{\theta} \cos y dy & 0 & \frac{1}{2} \int_{-\theta}^{\theta} \sin y dy \\ 0 & \frac{1}{2} \int_{-\theta}^{\theta} dy & 0 \\ -\frac{1}{2} \int_{-\theta}^{\theta} \sin y dy & 0 & \frac{1}{2} \int_{-\theta}^{\theta} \cos y dy \end{pmatrix} = \begin{pmatrix} \sin \theta & 0 & 0 \\ 0 & \theta & 0 \\ 0 & 0 & \sin \theta \end{pmatrix}$$

and **cosj**

$$\mathbf{cosj} \theta = \frac{1}{2} \frac{d}{d\theta} \int_{-\theta}^{\theta} e^{y \hat{j} \times} dy = \frac{d}{d\theta} \begin{pmatrix} \sin \theta & 0 & 0 \\ 0 & \theta & 0 \\ 0 & 0 & \sin \theta \end{pmatrix} = \begin{pmatrix} \frac{d}{d\theta} \sin \theta & 0 & 0 \\ 0 & \frac{d}{d\theta} \theta & 0 \\ 0 & 0 & \frac{d}{d\theta} \sin \theta \end{pmatrix} = \begin{pmatrix} \cos \theta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cos \theta \end{pmatrix}$$

For the yz -plane, for **sini**

$$\mathbf{sini} \theta = \frac{1}{2} \int_{-\theta}^{\theta} e^{y \hat{i} \times} dy = \frac{1}{2} \int_{-\theta}^{\theta} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos y & -\sin y \\ 0 & \sin y & \cos y \end{pmatrix} dy = \begin{pmatrix} \frac{1}{2} \int_{-\theta}^{\theta} dy & 0 & 0 \\ 0 & \frac{1}{2} \int_{-\theta}^{\theta} \cos y dy & -\frac{1}{2} \int_{-\theta}^{\theta} \sin y dy \\ 0 & \frac{1}{2} \int_{-\theta}^{\theta} \sin y dy & \frac{1}{2} \int_{-\theta}^{\theta} \cos y dy \end{pmatrix} = \begin{pmatrix} \theta & 0 & 0 \\ 0 & \sin \theta & 0 \\ 0 & 0 & \sin \theta \end{pmatrix}$$

and **cosi**

$$\mathbf{cosi} \theta = \frac{1}{2} \frac{d}{d\theta} \int_{-\theta}^{\theta} e^{y \hat{i} \times} dy = \frac{d}{d\theta} \begin{pmatrix} \theta & 0 & 0 \\ 0 & \sin \theta & 0 \\ 0 & 0 & \sin \theta \end{pmatrix} = \begin{pmatrix} \frac{d}{d\theta} \theta & 0 & 0 \\ 0 & \frac{d}{d\theta} \sin \theta & 0 \\ 0 & 0 & \frac{d}{d\theta} \sin \theta \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & 0 \\ 0 & 0 & \cos \theta \end{pmatrix}$$

k -cross in terms of Jordan matrices

This next bit of maths, which regards Jordan matrices, is probably not all that relevant [for this document], although will probably turn out to be very relevant in some way in the future [are now, with the nilpotent operators such as $i \times j \times$ for example]. More knowledge is always better than less knowledge. Jordan matrices have applications for solving differential equations with repeated roots, so this will be the important application whereby this theory applied to Jordan forms will have relevance [i.e., in documents where there are derivatives rather than a focus here on pure vector and matrix algebra]. So as a mathematical curiosity [for the time being], diadically expanded $i = \hat{k} \times$ can be written in terms of Jordan matrices. The Jordan matrix of relevance here is the direct sum $\oplus$ two Jordan blocks, that of a 2by2 Jordan matrix $J_2(\lambda)$, of arbitrary parameter, and a 1by1 trivial Jordan matrix $J_1(\lambda) = \lambda$

$$J_2(\lambda) = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \quad J_1(\lambda) = (\lambda)$$

The direct sum is then

$$J_2(\lambda) \oplus J_1(\lambda) = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

with transpose

$$(J_2(\lambda) \oplus J_1(\lambda))^T = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}^T = \begin{pmatrix} \lambda & 0 & 0 \\ 1 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} = J_2(\lambda)^T \oplus J_1(\lambda)$$

k -cross is then the (neg.) difference between the Jordan matrix and its transpose

$$\begin{aligned} J_2(\lambda)^T \oplus J_1(\lambda) - J_2(\lambda) \oplus J_1(\lambda) &= \begin{pmatrix} \lambda & 0 & 0 \\ 1 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} - \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \\ &= \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \hat{k} \times \end{aligned}$$

Also, for the k -hat exponential **expk** at $\pi/2$, which retains the rotational eigenvalue which differentiates it from k -cross (full rank rather than singular),

$$\begin{aligned}
(J_2(\lambda)^T - J_2(\lambda)) \oplus J_1(1) &= \left(\begin{pmatrix} \lambda & 0 \\ 1 & \lambda \end{pmatrix} - \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \right) \oplus J_1(1) \\
&= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \oplus J_1(1) \\
&= \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \exp(\pi \hat{k} \times (\cdot) / 2) \\
&= e^{0.5\pi \hat{k} \times}(\cdot)
\end{aligned}$$

Hyperbolic Basis Sines and Cosines: Double Cross, $i^2 = \hat{k} \times \hat{k} \times$

For second order single component cross products, there is a single double cross for each rotation axis and consequent basis rotation operator $\hat{k} \times$, $\hat{j} \times$, and $\hat{i} \times$. These are the k -hat double cross, the j -hat double cross, and the i -hat double cross: $\hat{k} \times \hat{k} \times$, $\hat{j} \times \hat{j} \times$, and $\hat{i} \times \hat{i} \times$. These are the basis operator equivalent to $i^2 = -1$. These operators covert to coordinate-system-dependent hyperbolic sines and cosines within the context of Euler's theorem and then recombine into scalar exponentials. There are three of each hyperbolic sine and hyperbolic cosine and the scalars sinh and cosh. Exponentials of single basis curls represent **harmonically curled vectors** whilst those of double basis curls (the double crosses) represent **hyperbolic divergence vectors**. The double cross is the square of complex i in the context of basis curls [basis cross products] as operators on the component basis-vectors. The trigonometric functions that are functions of these second order basis curls (the double crosses), which are hence basis-hat operators, are labelled **coskk**, **cosjj**, **cosii** (“cos-kay-kay”, “cos-jay-jay”, and “cos-eye-eye”) for the cosines and **sinkk**, **sinjj**, and **sinii** (“sin-kay-kay”, “sin-jay-jay”, and “sin-eye-eye”) for the sines. There are similarly the scalar hyperbolic cosine and sine, cosh and sinh, to which the double-cross basis-hat trigonometric (hyperbolic) operators revert after operating on a basis variable, that is, when they are not defined in their own directions. When they are defined in their own directions they degenerate, similarly as the single basis-hat trigonometric functions, to their own the eigenvectors and unity eigenvalue. These do not spin around an axis but diverge away from it (from a source) or into it (into a sink). Note that given a **double cross** is a bit ominous sounding, these can equivalently be called the **double curled k -hat, j -hat, and i -hat functions** (although is not quite as good because these functions diverge rather than curl). **Squared k -cross** might be a good, less ominous, term for the **double cross**. Divergence/convergence hyperbolics, or reflection hyperbolics is possible nomenclature, given that the double crosses are equivalent to group theory reflections ($i^2 = -1$) and generate roto-reflections or rotation-with-damping when coupled with **expk**, **expj**, and **expi**. Note importantly that the double crosses (of k -hat, j -hat, and i -hat) is why the **curl of curl** diverges (as radiation [photons] for example in Maxwell's equations which diverges [for both $\mathbf{E}$ and $\mathbf{B}$] as a Laplacian after curling the curl equations [into a photon which is then a ripple in the $\mathbf{E}$ - $\mathbf{B}$ basis]).

Considering the $\hat{k} \times \hat{k} \times$ cosine (the double cross k -hat cosine) **coskk**. Expressing the $\hat{k} \times \hat{k} \times$ dependent cosine in terms of the double cross k -hat exponentials

$$\mathbf{coskk}\theta = \cosh(\theta \hat{k} \times \hat{k} \times (\cdot)) = \cosh(\theta \hat{k} \times \hat{k} \times) = \frac{1}{2}(e^{\theta \hat{k} \times \hat{k} \times} + e^{-\theta \hat{k} \times \hat{k} \times})$$

Expanding in a power series

$$\begin{aligned}
\mathbf{cosk}k\theta &= \frac{1}{2}(e^{\theta \hat{k} \times \hat{k} \times} + e^{-\theta \hat{k} \times \hat{k} \times}) = \frac{1}{2}e^{\theta \hat{k} \times \hat{k} \times} + \frac{1}{2}e^{-\theta \hat{k} \times \hat{k} \times} \\
&= \frac{1}{2}\left(1 + \theta(\hat{k} \times \hat{k} \times) + \frac{1}{2!}\theta^2(\hat{k} \times \hat{k} \times)^2 + \frac{1}{3!}\theta^3(\hat{k} \times \hat{k} \times)^3 + \frac{1}{4!}\theta^4(\hat{k} \times \hat{k} \times)^4 + \dots\right) \\
&\quad + \frac{1}{2}\left(1 - \theta(\hat{k} \times \hat{k} \times) + \frac{1}{2!}\theta^2(\hat{k} \times \hat{k} \times)^2 - \frac{1}{3!}\theta^3(\hat{k} \times \hat{k} \times)^3 + \frac{1}{4!}\theta^4(\hat{k} \times \hat{k} \times)^4 + \dots\right) \\
&= 1 + \frac{1}{2!}\theta^2(\hat{k} \times)^4 + \frac{1}{4!}\theta^4(\hat{k} \times)^8
\end{aligned}$$

When it is understood that the cosine is associated with $\hat{i}$ or $\hat{j}$, so that $(\hat{k} \times)^4(\cdot) = (-1)^2(\cdot) = 1(\cdot)$ then

$$\mathbf{cosk}k\theta = \cosh\left(\theta \hat{k} \times \hat{k} \times (\cdot)\right) = \cosh(\theta \hat{k} \times \hat{k} \times) = 1 + \frac{1}{2!}\theta^2 + \frac{1}{4!}\theta^4 + \dots$$

which is the power series expansion of hyperbolic cosine and algebraically independent of $\hat{k} \times \hat{k} \times (\cdot) = -1(\cdot)$. But it is understood that these are operators. So, here, the double-cross k -hat cosine (the $\hat{k} \times \hat{k} \times$ cosine) is always associated with a direction, either $\hat{i}$ or $\hat{j}$, so like **cosk**, any power of $\hat{k} \times \hat{k} \times (\cdot) = -1(\cdot)$ does not have meaning outside of a coupling with a direction. In the $\hat{i}$ direction

$$(\hat{k} \times)^4 \hat{i} = \hat{k} \times \hat{k} \times \hat{k} \times (\hat{k} \times \hat{i}) = \hat{k} \times \hat{k} \times (\hat{k} \times \hat{j}) = \hat{k} \times \hat{k} \times (-\hat{i}) = -\hat{k} \times (\hat{k} \times \hat{i}) = -\hat{k} \times (\hat{j}) = \hat{i}, \text{ that is, } (\hat{k} \times)^4 \hat{i} = \hat{i} \text{ so that}$$

$$\begin{aligned}
\mathbf{cosk}k\theta \hat{i} &= \left(1 + \frac{1}{2!}\theta^2(\hat{k} \times)^4 + \frac{1}{4!}\theta^4(\hat{k} \times)^8 + \dots\right) \hat{i} \\
&= \hat{i} + \frac{1}{2!}\theta^2(\hat{k} \times)^4 \hat{i} + \frac{1}{4!}\theta^4(\hat{k} \times)^8 \hat{i} + \dots \\
&= \hat{i} + \frac{1}{2!}\theta^2 \hat{i} + \frac{1}{4!}\theta^4 \hat{i} - \dots \\
&= \left(1 + \frac{1}{2!}\theta^2 + \frac{1}{4!}\theta^4 + \dots\right) \hat{i} \\
&= \hat{i} \left(1 + \frac{1}{2!}\theta^2 + \frac{1}{4!}\theta^4 + \dots\right) \\
&= \hat{i} \cosh \theta
\end{aligned}$$

since $\left(1 + \frac{1}{2!}\theta^2 + \frac{1}{4!}\theta^4 + \dots\right) = \cosh \theta$ is scalar and hence commutes with i -hat

which is the scalar hyperbolic cosine defined on the i -hat axis. Similarly for the $\hat{k} \times \hat{k} \times$ cosine in the $\hat{j}$ direction

$$\begin{aligned}
\mathbf{cosk}k\theta \hat{j} &= \left(1 + \frac{1}{2!}\theta^2(\hat{k} \times)^4 + \frac{1}{4!}\theta^4(\hat{k} \times)^8 + \dots\right) \hat{j} \\
&= \hat{j} + \frac{1}{2!}\theta^2(\hat{k} \times)^4 \hat{j} + \frac{1}{4!}\theta^4(\hat{k} \times)^8 \hat{j} + \dots \\
&= \hat{j} + \frac{1}{2!}\theta^2 \hat{j} + \frac{1}{4!}\theta^4 \hat{j} - \dots \\
&= \left(1 + \frac{1}{2!}\theta^2 + \frac{1}{4!}\theta^4 + \dots\right) \hat{j} \\
&= \hat{j} \left(1 + \frac{1}{2!}\theta^2 + \frac{1}{4!}\theta^4 + \dots\right) \\
&= \hat{j} \cosh \theta
\end{aligned}$$

since $\left(1 + \frac{1}{2!}\theta^2 + \frac{1}{4!}\theta^4 + \dots\right) = \cosh \theta$ is scalar and hence commutes with j -hat

since $(\hat{k} \times)^4 \hat{j} = \hat{k} \times \hat{k} \times \hat{k} \times (\hat{k} \times \hat{j}) = \hat{k} \times \hat{k} \times \hat{k} \times (-\hat{i}) = -\hat{k} \times \hat{k} \times (\hat{k} \times \hat{i}) = -\hat{k} \times (\hat{k} \times \hat{j}) = -\hat{k} \times (-\hat{i}) = \hat{j}$, that is, $(\hat{k} \times)^4 \hat{j} = \hat{j}$. This is the scalar hyperbolic cosine defined on the j -hat axis.

Similarly as the $\hat{k} \times$ cosine, when the $\hat{k} \times \hat{k} \times$ is associated with the $\hat{k}$ direction, all terms in $\hat{k} \times \hat{k} \times$ zero and the **hyperbolic properties degenerate**. This then results in the small angle approximation to cosh, which in this context is its direction on the $\hat{k}$ axis. This is the eigenvalue associated with the unitary eigenvector of **cosk**, which is k -hat

$$\begin{aligned}
\text{coskk}\theta \hat{k} &= \left(1 + \frac{1}{2!}\theta^2(\hat{k} \times)^4 + \frac{1}{4!}\theta^4(\hat{k} \times)^8 + \dots\right) \hat{k} \\
&= \hat{k} + \frac{1}{2!}\theta^2(\hat{k} \times)^4 \hat{k} + \frac{1}{4!}\theta^4(\hat{k} \times)^8 \hat{k} + \dots \\
&= \hat{k} + \frac{1}{2!}\theta^2(0\hat{k}) + \frac{1}{4!}\theta^4(0\hat{k}) - \dots \\
&= \left(1 + \frac{1}{2!}\theta^2(0) + \frac{1}{4!}\theta^4(0) + \dots\right) \hat{k} \\
&= (1 + 0 + 0 + 0 + 0 + \dots) \hat{k} \\
&= \hat{k}(1 + 0 + 0 + 0 + 0 + \dots) \\
&= \hat{k}
\end{aligned}$$

This is the eigenvector of the double-cross divergence.

These results have hence demonstrated that there are three different hyperbolic sine and three different hyperbolic cosine operators as functions of the double cross (as well as the scalar hyperbolics), one each associated with their basis-hat: double cross k -hat cosine (**coskk**), double cross j -hat cosine (**cosjj**), double cross i -hat cosine (**cosii**). Again, these are operators and once operated on a basis-hat, become scalar hyperbolic or scalar constant.

In summary, for the double-cross k -hat cosine (the $\hat{k} \times \hat{k} \times$ cosine), those associated with $\exp(\theta \hat{k} \times \hat{k} \times (\cdot)) = \exp(\theta \hat{k} \times \hat{k} \times)$, coupled with each direction are

$$\begin{aligned}
(\text{coskk}\theta) \hat{k} &= (1 + 0 + 0 + \dots) \hat{k} = \hat{k}(1 + 0 + 0 + \dots) = \hat{k} && \text{("cos-kay-kay theta kay-hat")} \\
(\text{coskk}\theta) \hat{j} &= \left(1 + \frac{1}{2!}\theta^2 + \frac{1}{4!}\theta^4 + \dots\right) \hat{j} = \hat{j} \left(1 + \frac{1}{2!}\theta^2 + \frac{1}{4!}\theta^4 + \dots\right) = \hat{j} \cosh \theta && \text{("cos-kay-kay theta jay-hat")} \\
(\text{coskk}\theta) \hat{i} &= \left(1 + \frac{1}{2!}\theta^2 + \frac{1}{4!}\theta^4 + \dots\right) \hat{i} = \hat{i} \left(1 + \frac{1}{2!}\theta^2 + \frac{1}{4!}\theta^4 + \dots\right) = \hat{i} \cosh \theta && \text{("cos-kay-kay theta eye-hat")}
\end{aligned}$$

cos-kay-kay theta kay-hat (before basis-hat operation) = **kay-hat** (after basis-hat operation)

cos-kay-kay theta jay-hat (before basis-hat operation) = **jay-hat cosh theta** (after basis-hat operation)

cos-kay-kay theta eye-hat (before basis-hat operation) = **eye-hat cosh theta** (after basis-hat operation)

Considering the $\hat{k} \times \hat{k} \times$ sine (the double cross k -hat sine **sinkk**). Expressing the $\hat{k} \times \hat{k} \times$ dependent sine in terms of the double cross k -hat exponentials

$$\text{sinkk}\theta = \left(\hat{k} \times \hat{k} \times (\cdot)\right)^{-1} \sinh\left(\theta \hat{k} \times \hat{k} \times (\cdot)\right) = (\hat{k} \times \hat{k} \times)^{-1} \sinh(\theta \hat{k} \times \hat{k} \times) = (\hat{k} \times \hat{k} \times)^{-1} \frac{1}{2} (e^{\theta \hat{k} \times \hat{k} \times} - e^{-\theta \hat{k} \times \hat{k} \times}) = (\hat{k} \times \hat{k} \times)^{-1} \left(\frac{1}{2}(e^{\theta \hat{k} \times \hat{k} \times}) - \frac{1}{2}(e^{-\theta \hat{k} \times \hat{k} \times})\right)$$

Expanding the double cross k -hat sine (the $\hat{k} \times \hat{k} \times$ sine) in a power series

$$\begin{aligned}
\text{sinkk}\theta &= (\hat{k} \times \hat{k} \times)^{-1} \left(\frac{1}{2}(e^{\theta \hat{k} \times \hat{k} \times}) - \frac{1}{2}(e^{-\theta \hat{k} \times \hat{k} \times})\right) \\
&= (\hat{k} \times \hat{k} \times)^{-1} \left(\frac{1}{2}(1 + \theta(\hat{k} \times \hat{k} \times) + \frac{1}{2!}\theta^2(\hat{k} \times \hat{k} \times)^2 + \frac{1}{3!}\theta^3(\hat{k} \times \hat{k} \times)^3 + \frac{1}{4!}\theta^4(\hat{k} \times \hat{k} \times)^4 + \dots) \right. \\
&\quad \left. - \frac{1}{2}(1 - \theta(\hat{k} \times \hat{k} \times) + \frac{1}{2!}\theta^2(\hat{k} \times \hat{k} \times)^2 - \frac{1}{3!}\theta^3(\hat{k} \times \hat{k} \times)^3 + \frac{1}{4!}\theta^4(\hat{k} \times \hat{k} \times)^4 + \dots) \right) \\
&= (\hat{k} \times \hat{k} \times)^{-1} \left(\theta(\hat{k} \times \hat{k} \times) + \frac{1}{3!}\theta^3(\hat{k} \times \hat{k} \times)^3 + \frac{1}{5!}\theta^5(\hat{k} \times \hat{k} \times)^5 + \dots\right) \\
&= \theta + \frac{1}{3!}\theta^3(\hat{k} \times \hat{k} \times)^2 + \frac{1}{5!}\theta^5(\hat{k} \times \hat{k} \times)^4 + \dots
\end{aligned}$$

where in the circumstances that the hyperbolic sine is associated with $\hat{i}$ or $\hat{j}$, then $(\hat{k} \times \hat{k} \times)^2 = 1$. So the expansion is

$$\theta + \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 - \dots = \sinh \theta$$

which is the standard power series expansion of hyperbolic sine. So in either the i -hat or j -hat direction, the double-cross k -hat sine **sinkk** is scalar hyperbolic sine. That is, the double-cross k -hat sine **sinkk** reverts to hyperbolic sine after operating on either i -hat or j -hat. In the i -hat direction

$$\begin{aligned} (\mathbf{sinkk} \theta) \hat{i} &= (\hat{k} \times \hat{k} \times)^{-1} \left(\theta (\hat{k} \times \hat{k} \times) + \frac{1}{3!} \theta^3 (\hat{k} \times \hat{k} \times)^3 + \frac{1}{5!} \theta^5 (\hat{k} \times \hat{k} \times)^5 + \dots \right) \hat{i} \\ &= \left(\theta + \frac{1}{3!} \theta^3 (\hat{k} \times \hat{k} \times)^2 + \frac{1}{5!} \theta^5 (\hat{k} \times \hat{k} \times)^4 + \dots \right) \hat{i} \\ &= \theta \hat{i} + \frac{1}{3!} \theta^3 (\hat{k} \times \hat{k} \times)^2 \hat{i} + \frac{1}{5!} \theta^5 (\hat{k} \times \hat{k} \times)^4 \hat{i} + \dots \\ &= \theta \hat{i} + \frac{1}{3!} \theta^3 \hat{i} + \frac{1}{5!} \theta^5 \hat{i} + \dots \\ &= \left(\theta + \frac{1}{3!} \theta^3 + \frac{1}{5!} \theta^5 + \dots \right) \hat{i} \\ &= \hat{i} \left(\theta + \frac{1}{3!} \theta^3 + \frac{1}{5!} \theta^5 + \dots \right) \\ &= \hat{i} \sinh \theta \end{aligned}$$

since $\left(\theta + \frac{1}{3!} \theta^3 + \frac{1}{5!} \theta^5 + \dots \right) = \sinh \theta$ is scalar and hence commutes with $\hat{i}$. In the j -hat direction

$$\begin{aligned} \mathbf{sinkk} \theta \hat{j} &= \left(\theta + \frac{1}{3!} \theta^3 + \frac{1}{5!} \theta^5 + \dots \right) \hat{j} \\ &= \hat{j} \left(\theta + \frac{1}{3!} \theta^3 + \frac{1}{5!} \theta^5 + \dots \right) \\ &= \hat{j} \sinh \theta \end{aligned}$$

For the $\hat{k} \times \hat{k} \times$ sine, **sinkk**, in the $\hat{k}$ direction (its own basis rotation axis), all of the powers of $\hat{k} \times \hat{k} \times$ zero so that the hyperbolic properties degenerate upon coupling with, and operating on, the k -hat basis variable. The outcome for this component is the same as the single cross k -hat sine **sink**, that is, degenerates into the small angle approximation for sinh (which is the same as that for trigonometric sine) in the $\hat{k}$ direction

$$\begin{aligned} \mathbf{sinkk} \theta \hat{k} &= (\hat{k} \times \hat{k} \times)^{-1} \left(\theta (\hat{k} \times \hat{k} \times) + \frac{1}{3!} \theta^3 (\hat{k} \times \hat{k} \times)^3 + \frac{1}{5!} \theta^5 (\hat{k} \times \hat{k} \times)^5 + \dots \right) \hat{k} \\ &= \left(\theta + \frac{1}{3!} \theta^3 (\hat{k} \times \hat{k} \times)^2 + \frac{1}{5!} \theta^5 (\hat{k} \times \hat{k} \times)^4 + \dots \right) \hat{k} \\ &= \theta \hat{k} + \frac{1}{3!} \theta^3 (\hat{k} \times)^4 \hat{k} + \frac{1}{5!} \theta^5 (\hat{k} \times)^6 \hat{k} + \dots \\ &= \theta \hat{k} + \frac{1}{3!} \theta^3 (0 \hat{k}) + \frac{1}{5!} \theta^5 (0 \hat{k}) + \dots \\ &= \left(\theta \hat{k} + \frac{1}{3!} \theta^3 (0) + \frac{1}{5!} \theta^5 (0) + \dots \right) \hat{k} \\ &= (\theta + 0 + 0 + \dots) \hat{k} \\ &= \hat{k} (\theta + 0 + 0 + \dots) \\ &= \theta \hat{k} \end{aligned}$$

In summary, for the double cross k -hat sine (the $\hat{k} \times \hat{k} \times$ sine)

$$(\text{sinkk}\theta)\hat{k} = (\hat{k} \times)^{-1} \sinh(\theta \hat{k} \times \hat{k} \times) \hat{k} = (\theta + 0 + 0 + \dots) \hat{k} = \hat{k}(\theta + 0 + 0 + \dots) = \hat{k} \theta \quad (\text{"sine-kay-kay theta kay-hat"})$$

$$(\text{sinkk}\theta)\hat{j} = (\hat{k} \times \hat{k} \times)^{-1} \sinh(\theta \hat{k} \times \hat{k} \times) \hat{j} = (\theta + \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 + \dots) \hat{j} = \hat{j}(\theta + \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 + \dots) = \hat{j} \sinh \theta \quad (\text{"sine-kay-kay theta jay-hat"})$$

$$(\text{sinkk}\theta)\hat{i} = (\hat{k} \times \hat{k} \times)^{-1} \sinh(\theta \hat{k} \times \hat{k} \times) \hat{i} = (\theta + \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 + \dots) \hat{i} = \hat{i}(\theta + \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 + \dots) = \hat{i} \sinh \theta \quad (\text{"sine-kay-kay theta eye-hat"})$$

sine-kay-kay theta kay-hat (before basis-hat operation) = **kay-hat theta** (after basis-hat operation)

sine-kay-kay theta jay-hat (before basis-hat operation) = **jay-hat sinh theta** (after basis-hat operation)

sine-kay-kay theta eye-hat (before basis-hat operation) = **eye-hat sinh theta** (after basis-hat operation)

Table 7 Summary: k-hat, j-hat, and i-hat double cross sines and cosines

Scalar function	k-hat operators	j-hat operators	i-hat operators
$\sinh \theta$	$(\text{sinkk} \theta) \hat{k} = \hat{k} \theta$	$(\text{sinjj} \theta) \hat{k} = \hat{k} \sinh \theta$	$(\text{sinii} \theta) \hat{k} = \hat{k} \sinh \theta$
	$(\text{sinkk} \theta) \hat{j} = \hat{j} \sinh \theta$	$(\text{sinjj} \theta) \hat{j} = \hat{j} \theta$	$(\text{sinii} \theta) \hat{j} = \hat{j} \sinh \theta$
	$(\text{sinkk} \theta) \hat{i} = \hat{i} \sinh \theta$	$(\text{sinjj} \theta) \hat{i} = \hat{i} \sinh \theta$	$(\text{sinii} \theta) \hat{i} = \hat{i} \theta$
$\cosh \theta$	$(\text{coskk} \theta) \hat{k} = \hat{k}$	$(\text{cosjj} \theta) \hat{k} = \hat{k} \cosh \theta$	$(\text{cosii} \theta) \hat{k} = \hat{k} \cosh \theta$
	$(\text{coskk} \theta) \hat{j} = \hat{j} \cosh \theta$	$(\text{cosjj} \theta) \hat{j} = \hat{j}$	$(\text{cosii} \theta) \hat{j} = \hat{j} \cosh \theta$
	$(\text{coskk} \theta) \hat{i} = \hat{i} \cosh \theta$	$(\text{cosjj} \theta) \hat{i} = \hat{i} \cosh \theta$	$(\text{cosii} \theta) \hat{i} = \hat{i}$

These are very similar to those of the scalar hyperbolic functions, especially when diadically expanded for the basis-hat exponential. So how would one know whether the function required is a double cross basis-hat or just an ordinary scalar. I don't know yet but the main reason would be that the double cross basis-hat operators are tied to a coordinate system and hence have the equivalent of a rotation matrix but for a divergence, here a damping or reflection. Quadruple crosses don't add much and triple-crosses are similar to the single crosses so I don't think this will be that much of a point of consternation or an uncertain decision point. I think the main question will be 'does the function curl or diverge or both?' The double crosses can interact with single crosses and then produce spirals, that is, a rotation into a sink (exponentially damped circles) or a rotation out of a source. This is without a single bit of calculus but I am expecting there to be a one-for-one correspondence [that is, when I get started on that project if I have any computing that still works]. Note also that $\hat{k} \times \hat{k} \times$ isn't invertible since it is singular similar as $\hat{k} \times$. In most circumstances it suffices to just use $(\text{sinkk} \theta) \hat{k} \times \hat{k} \times$ as the operator (as opposed to **sinkk** without the trailing $\hat{k} \times \hat{k} \times$) as considered now in hyperbolic exponential form which is that expected to be most relevant for double-crosses, i.e., for reflections, divergences, and damping.

Hyperbolic Basis Exponential – the double cross ($i^2 = \hat{k} \times \hat{k} \times$) k-hat exponential

The power series expansion of the double-cross k-hat exponential, with an unspecified direction (in the i-hat or j-hat direction), is

$$\begin{aligned}
 e^{\theta \hat{k} \times \hat{k} \times} &= \exp(\theta \hat{k} \times \hat{k} \times (\cdot)) \\
 &= 1 + \theta(\hat{k} \times \hat{k} \times) + \frac{1}{2!}\theta^2(\hat{k} \times \hat{k} \times)^2 + \frac{1}{3!}\theta^3(\hat{k} \times \hat{k} \times)^3 + \frac{1}{4!}\theta^4(\hat{k} \times \hat{k} \times)^4 + \dots \\
 &= 1 - \theta + \frac{1}{2!}\theta^2 - \frac{1}{3!}\theta^3 + \frac{1}{4!}\theta^4 - \dots
 \end{aligned}$$

Separating the odd and even components of the expansion is not as relevant to the double cross hyperbolic divergences since the exponential doesn't separate out onto two axes, that is, the cosh and sinh remain on the same basis-vector because of the double cross. Expanding and separating them anyway for the sake of cosh and sinh and the Euler expression

$$\begin{aligned}
 e^{\theta \hat{k} \times \hat{k} \times} &= \exp\left(\theta \hat{k} \times \hat{k} \times (\cdot)\right) \\
 &= \left(1 + \frac{1}{2!} \theta^2 + \frac{1}{4!} \theta^4 + \dots\right) - \left(\theta + \frac{1}{3!} \theta^3 + \frac{1}{5!} \theta^5 + \dots\right) \\
 &= \left(1 - \frac{1}{2!} \theta^2 + \frac{1}{4!} \theta^4 - \dots\right) - \left(\theta - \frac{1}{3!} \theta^3 + \frac{1}{5!} \theta^5 - \dots\right) \\
 &= \cosh \theta - \sinh \theta \\
 &= e^{-\theta}
 \end{aligned}$$

Since the cosine and sine have not yet been applied to a basis direction, these are still operators (latently) that depend on even powers of the double $\hat{k}$ -cross. More completely

$$\begin{aligned}
 e^{\theta \hat{k} \times \hat{k} \times} &= \exp\left(\theta \hat{k} \times \hat{k} \times (\cdot)\right) \\
 &= \left(1 - \frac{1}{2!} \theta^2 (\hat{k} \times \hat{k} \times)^2 + \frac{1}{4!} \theta^4 (\hat{k} \times \hat{k} \times)^4 - \dots\right) + \left(\theta (\hat{k} \times \hat{k} \times) - \frac{1}{3!} \theta^3 (\hat{k} \times \hat{k} \times)^3 + \frac{1}{5!} \theta^5 (\hat{k} \times \hat{k} \times)^5 - \dots\right) \\
 &= \left(1 - \frac{1}{2!} \theta^2 + \frac{1}{4!} \theta^4 - \dots\right) + \left(\theta - \frac{1}{3!} \theta^3 + \frac{1}{5!} \theta^5 - \dots\right) (\hat{k} \times \hat{k} \times) \\
 &= \mathbf{coskk} \theta + (\mathbf{sinkk} \theta) \hat{k} \times \hat{k} \times (\cdot)
 \end{aligned}$$

which is in the form of Euler's equation for diverging/converging damping basis hyperbolics. Specified in the x direction with i -hat

$$\begin{aligned}
 e^{\theta \hat{k} \times \hat{k} \times} \hat{i} &= \exp\left(\theta \hat{k} \times \hat{k} \times (\cdot)\right) \hat{i} \\
 &= \left(1 + \theta (\hat{k} \times \hat{k} \times) + \frac{1}{2!} \theta^2 (\hat{k} \times \hat{k} \times)^2 + \frac{1}{3!} \theta^3 (\hat{k} \times \hat{k} \times)^3 + \frac{1}{4!} \theta^4 (\hat{k} \times \hat{k} \times)^4 + \dots\right) \hat{i} \\
 &= \hat{i} + \theta (\hat{k} \times \hat{k} \times) \hat{i} + \frac{1}{2!} \theta^2 (\hat{k} \times \hat{k} \times)^2 \hat{i} + \frac{1}{3!} \theta^3 (\hat{k} \times \hat{k} \times)^3 \hat{i} + \frac{1}{4!} \theta^4 (\hat{k} \times \hat{k} \times)^4 \hat{i} + \dots \\
 &= \hat{i} - \theta \hat{i} + \frac{1}{2!} \theta^2 \hat{i} - \frac{1}{3!} \theta^3 \hat{i} + \frac{1}{4!} \theta^4 \hat{i} + \dots \\
 &= \hat{i} \left(1 - \theta + \frac{1}{2!} \theta^2 - \frac{1}{3!} \theta^3 + \frac{1}{4!} \theta^4 + \dots\right) \\
 &= \hat{i} e^{-\theta}
 \end{aligned}$$

that is,

$$\begin{aligned}
 e^{\theta \hat{k} \times \hat{k} \times} \hat{i} &= \exp\left(\theta \hat{k} \times \hat{k} \times (\cdot)\right) \hat{i} = \left(\mathbf{coskk} \theta + (\mathbf{sinkk} \theta) \hat{k} \times \hat{k} \times\right) \hat{i} = (\mathbf{coskk} \theta) \hat{i} + (\mathbf{sinkk} \theta) \hat{k} \times \hat{k} \times \hat{i} \\
 &= (\mathbf{coskk} \theta) \hat{i} - (\mathbf{sinkk} \theta) \hat{i} = \hat{i} (\cosh \theta) - \hat{i} (\sinh \theta) = \hat{i} ((\cosh \theta) - (\sinh \theta)) = \hat{i} e^{-\theta}
 \end{aligned}$$

so it all works as it should, the decaying exponential scalar in the i -hat direction. Note again that it does not split into two components since both the cosh and sinh remain on the same axis generated by i -hat (remain on the x -axis without a component cross over to that of the y -axis of j -hat). When specified in the y direction with j -hat the results are similar

$$\begin{aligned}
e^{\theta \hat{k} \times \hat{k} \times} \hat{j} &= \exp\left(\theta \hat{k} \times \hat{k} \times (\cdot)\right) \hat{j} \\
&= \left(1 + \theta(\hat{k} \times \hat{k} \times) + \frac{1}{2!} \theta^2 (\hat{k} \times \hat{k} \times)^2 + \frac{1}{3!} \theta^3 (\hat{k} \times \hat{k} \times)^3 + \frac{1}{4!} \theta^4 (\hat{k} \times \hat{k} \times)^4 + \dots\right) \hat{j} \\
&= \hat{j} + \theta(\hat{k} \times \hat{k} \times) \hat{j} + \frac{1}{2!} \theta^2 (\hat{k} \times \hat{k} \times)^2 \hat{j} + \frac{1}{3!} \theta^3 (\hat{k} \times \hat{k} \times)^3 \hat{j} + \frac{1}{4!} \theta^4 (\hat{k} \times \hat{k} \times)^4 \hat{j} + \dots \\
&= \hat{j} - \theta \hat{j} + \frac{1}{2!} \theta^2 \hat{j} - \frac{1}{3!} \theta^3 \hat{j} + \frac{1}{4!} \theta^4 \hat{j} + \dots \\
&= \hat{j} \left(1 - \theta + \frac{1}{2!} \theta^2 - \frac{1}{3!} \theta^3 + \frac{1}{4!} \theta^4 + \dots\right) \\
&= \hat{j} e^{-\theta}
\end{aligned}$$

which is again the same theta decaying exponential scalar but in the j -hat direction, i.e., the exact same hyperbolic divergence of the double cross k -hat exponential but in the j -hat (rather than i -hat) direction. Alternatively, for the same result (but alternate maths): when the double-cross k -hat exponential $e^{\theta \hat{k} \times \hat{k} \times} = \exp\left(\theta \hat{k} \times \hat{k} \times (\cdot)\right)$ is specified in the y -direction with $\hat{j}$ and is expressed terms of the hyperbolics then

$$\begin{aligned}
e^{\theta \hat{k} \times \hat{k} \times} \hat{j} &= \exp\left(\theta \hat{k} \times \hat{k} \times (\cdot)\right) \hat{j} = \left(\text{coskk} \theta + (\text{sinkk} \theta) \hat{k} \times \hat{k} \times\right) \hat{j} = (\text{coskk} \theta) \hat{j} + (\text{sinkk} \theta) \hat{k} \times \hat{k} \times \hat{j} \\
&= (\text{coskk} \theta) \hat{j} - (\text{sinkk} \theta) \hat{j} = \hat{j} (\cosh \theta) - \hat{j} (\sinh \theta) = \hat{j} ((\cosh \theta) - (\sinh \theta)) = \hat{j} e^{-\theta}
\end{aligned}$$

The double cross k -hat exponential in the k -hat direction (the double cross k -hat exponential pointing in its own direction) degenerates similarly as the single basis-cross k -hat exponential. The component hyperbolics degenerate so that it reverts to its own basis direction, that is, to the eigenvector of the divergence (rather than circulation):

$$\begin{aligned}
e^{\theta \hat{k} \times \hat{k} \times} \hat{k} &= \exp\left(\theta \hat{k} \times \hat{k} \times (\cdot)\right) \hat{k} = \left(\text{coskk} \theta + (\text{sinkk} \theta) \hat{k} \times \hat{k} \times\right) \hat{k} = (\text{coskk} \theta) \hat{k} + (\text{sinkk} \theta) \hat{k} \times \hat{k} \times \hat{k} = (\text{coskk} \theta) \hat{k} - (\text{sinkk} \theta) 0 \\
&= \hat{k}
\end{aligned}$$

In summary, there are three double-cross basis exponentials. These are the k -hat, j -hat, and i -hat double-cross basis exponentials **expkk**, **expjj**, and **expii**. These are divergences. de-Moivre's theorem is self-evident. These are most interesting in terms of coordinate systems which I will consider now.

Coordinate Systems – Diadic expansion of hyperbolic divergence of double-cross

For the double cross $\hat{k} \times \hat{k} \times$, in each direction of i -hat ($\hat{i}$), j -hat ($\hat{j}$), and k -hat ($\hat{k}$), this returns **(1)** their negative for i -hat and j -hat (as per the square of a complex number, $-\hat{i} = (-1)\hat{i}$ and $-\hat{j} = (-1)\hat{j}$) and **(2)** is zero for k -hat (in its own direction). This is

$$\begin{aligned}
(\hat{k} \times)^2 \hat{i} &= (\hat{k} \times \hat{k} \times) \hat{i} = -\hat{i} = (-1)\hat{i} \\
(\hat{k} \times)^2 \hat{j} &= (\hat{k} \times \hat{k} \times) \hat{j} = -\hat{j} = (-1)\hat{j} \\
(\hat{k} \times)^2 \hat{k} &= (\hat{k} \times \hat{k} \times) \hat{k} = 0 = 0 \hat{k}
\end{aligned}$$

which written in full is

$$(\hat{k} \times \hat{k} \times) \hat{i} = \hat{i}(-1) + \hat{j}(0) + \hat{k}(0)$$

$$(\hat{k} \times \hat{k} \times) \hat{j} = \hat{i}(0) + \hat{j}(-1) + \hat{k}(0)$$

$$(\hat{k} \times \hat{k} \times) \hat{k} = \hat{i}(0) + \hat{j}(0) + \hat{k}(0)$$

Each of these vectors are the columns of the diadically expanded matrix which is, for the double cross. So

$$\hat{k} \times \hat{k} \times (\cdot) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} (\cdot)$$

which can be compared with that of k -cross, it places (neg.) ones on the diagonal and zeroes the eigenvector of its own direction. For the double cross k -hat cosine, **coskk**, the components are

$$(\mathbf{coskk}\theta) \hat{i} = \hat{i} \cosh \theta + \hat{j}(0) + \hat{k}(0)$$

$$(\mathbf{coskk}\theta) \hat{j} = \hat{i}(0) + \hat{j} \cosh \theta + \hat{k}(0)$$

$$(\mathbf{coskk}\theta) \hat{k} = \hat{i}(0) + \hat{j}(0) + \hat{k}(1)$$

which diadically expands (components to columns) into

$$(\mathbf{coskk}\theta) (\cdot) = \begin{pmatrix} \cosh \theta & 0 & 0 \\ 0 & \cosh \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} (\cdot)$$

Similarly for double cross k -hat sine, **sinkk**; the components are

$$(\mathbf{sinkk}\theta) \hat{i} = \hat{i} \sinh \theta + \hat{j}(0) + \hat{k}(0)$$

$$(\mathbf{sinkk}\theta) \hat{j} = \hat{i}(0) + \hat{j} \sinh \theta + \hat{k}(0)$$

$$(\mathbf{sinkk}\theta) \hat{k} = \hat{i}(0) + \hat{j}(0) + \hat{k}(\theta)$$

which diadically expands into

$$(\mathbf{sinkk}\theta) (\cdot) = \begin{pmatrix} \sinh \theta & 0 & 0 \\ 0 & \sinh \theta & 0 \\ 0 & 0 & \theta \end{pmatrix} (\cdot)$$

Then for the double-cross k -hat exponential

$$\begin{aligned}
e^{\theta \hat{k} \times \hat{k} \times}(\cdot) &= \mathbf{cosk} \theta + (\mathbf{sink} \theta) \hat{k} \times \hat{k} \times (\cdot) = \begin{pmatrix} \cosh \theta & 0 & 0 \\ 0 & \cosh \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} \sinh \theta & 0 & 0 \\ 0 & \sinh \theta & 0 \\ 0 & 0 & \theta \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} \cosh \theta - \sinh \theta & 0 & 0 \\ 0 & \cosh \theta - \sinh \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} e^{-\theta} & 0 & 0 \\ 0 & e^{-\theta} & 0 \\ 0 & 0 & 1 \end{pmatrix} (\cdot)
\end{aligned}$$

which is an isotropic sink (basin) divergence on the x - y plane generated by i -hat and j -hat, the divergence parameterised by θ . The k -hat component for the k 's matrix basis entry is the eigenvector of the divergence. Unlike the rotation matrices, the divergence matrices of double-cross exponentials are diagonal. This means that they commute and can be combined in any order in any number of ways and they will simplify multiplicatively to another diagonal divergence matrix. This means that an isotropic planar divergence (expanding circular wavefronts) is easily extended to an isotropic volumetric divergence (expanding spherical wavefronts) or a polarised volumetric divergence (expanding ellipsoidal wavefront). These can also be mixed for source divergences with sink divergences on the different axes to produce shearing effects. When planar, these are saddles and the like.

Diadically expanding the components of the double cross k -hat exponential, the components for which are

$$\begin{aligned}
e^{\theta \hat{k} \times \hat{k} \times} \hat{i} &= \hat{i} e^{-\theta} + \hat{j} (0) + \hat{k} (0) \\
e^{\theta \hat{k} \times \hat{k} \times} \hat{j} &= \hat{i} (0) + \hat{j} e^{-\theta} + \hat{k} (0) \\
e^{\theta \hat{k} \times \hat{k} \times} \hat{k} &= \hat{i} (0) + \hat{j} (0) + \hat{k} (1)
\end{aligned}$$

the diadic vector expansion is the same divergence matrix (rather than rotation matrix) as before

$$e^{\theta \hat{k} \times \hat{k} \times}(\cdot) = \begin{pmatrix} e^{-\theta} & 0 & 0 \\ 0 & e^{-\theta} & 0 \\ 0 & 0 & 1 \end{pmatrix} (\cdot)$$

For the double cross j -hat exponential and i -hat exponential, the divergence matrices are

$$e^{\theta \hat{j} \times \hat{j} \times}(\cdot) = \begin{pmatrix} e^{-\theta} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-\theta} \end{pmatrix} (\cdot) \quad \text{and} \quad e^{\theta \hat{i} \times \hat{i} \times}(\cdot) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-\theta} & 0 \\ 0 & 0 & e^{-\theta} \end{pmatrix} (\cdot)$$

These are similarly isotropic sink (basin) divergences (actually convergence) on the x - z and y - z planes respectively. Changing the sign in the double cross basis-hat exponential changes the divergence (convergence) to a source divergence (as per the Gauss divergence theorem for a positive charge). These are radial lines for which the wavefront of the density of lines expand or contract as circles. So these matrices, as operators, have a unitary eigenvector for the divergence, similarly as the single basis-cross exponential for the circulation. The product of two rotational matrices (rather than divergence matrices) on orthogonal axes produces polar sphericals whilst the product of three rotational matrices on the three orthogonal axes is not as intuitive for what is known and common in mathematics but corresponds to a three rotational degree of freedom system of spin, precession, and nutation. The product two

divergence matrices is not easy to describe but the product of the three mutual divergence matrices, for a common argument, is a spherical divergence, that is, an isotropic sphere of divergence from a point source or into a point sink which is shown here. This product is

$$e^{\theta \hat{k} \times \hat{k} \times} e^{\theta \hat{j} \times \hat{j} \times} e^{\theta \hat{i} \times \hat{i} \times} (\cdot) = \begin{pmatrix} e^{-\theta} & 0 & 0 \\ 0 & e^{-\theta} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-\theta} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-\theta} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-\theta} & 0 \\ 0 & 0 & e^{-\theta} \end{pmatrix} = \begin{pmatrix} e^{-2\theta} & 0 & 0 \\ 0 & e^{-2\theta} & 0 \\ 0 & 0 & e^{-2\theta} \end{pmatrix} (\cdot)$$

or alternatively with independent arguments $x, y,$ and z for each axis (generated by i -hat, j -hat, k -hat respectively)

$$e^{z \hat{k} \times \hat{k} \times + y \hat{j} \times \hat{j} \times + x \hat{i} \times \hat{i} \times} (\cdot) = e^{z \hat{k} \times \hat{k} \times} e^{y \hat{j} \times \hat{j} \times} e^{x \hat{i} \times \hat{i} \times} = \begin{pmatrix} e^{-z} & 0 & 0 \\ 0 & e^{-z} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-y} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-y} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-x} & 0 \\ 0 & 0 & e^{-x} \end{pmatrix} = \begin{pmatrix} e^{-(y+z)} & 0 & 0 \\ 0 & e^{-(z+x)} & 0 \\ 0 & 0 & e^{-(x+y)} \end{pmatrix} (\cdot)$$

which is an operator that makes the operand spatially damped.

So the third order product of k -hat, j -hat, and i -hat for a common attenuation generate isotropic divergences (when isotropic in decay constants that is) from a point source, that is, radial lines emanating from a source whose density of lines decays by the inverse square law because of the one-on- R -square dependence of the spherical wavefront (as per the Gauss divergence). When the attenuation constants are not isotropic, the divergence will be polarised.

Note that if the double cross (squared k -cross) as a sink divergence is mathematically what gravity is, then this could be an explanation as to why planets and stars are spheres. If the sink divergence pulls matter isotropically in to the sink (centre of mass of a mass system), then matter would agglomerate cumulatively into a sphere.

Divergences and Curls (double k -cross with single k -cross) – Damped Rotation Matrices

Now combining the single crosses with the double crosses. The product of a single basis-curl exponential **expk** with that of a double-cross exponential **expkk** for the common basis vector, here for k -hat, produces a damped rotation around the z axis generated by k -hat which is a spiral into an isotropic basis sink (a circulation sink convergence)

$$e^{\alpha \hat{k} \times (\hat{k} \times) + \omega \hat{k} \times} (\cdot) = e^{\alpha \hat{k} \times} e^{\omega \hat{k} \times} = \begin{pmatrix} \cos \omega t & -\sin \omega t & 0 \\ \sin \omega t & \cos \omega t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-\alpha t} & 0 & 0 \\ 0 & e^{-\alpha t} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{-\alpha t} \cos \omega t & -e^{-\alpha t} \sin \omega t & 0 \\ e^{-\alpha t} \sin \omega t & e^{-\alpha t} \cos \omega t & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which has attenuation constant α and radial frequency ω . The isometric expressions for j -hat and i -hat are similar as per the positions of the relevant rotation matrices for the isometric axes. For **(1)** the j -hat exponential **expj** with the double cross j -hat exponential **expjj** and **(2)** the i -hat exponential **expi** with double cross i -hat exponential **expii** are

$$\textbf{(1)} \quad e^{\alpha \hat{j} \times (\hat{j} \times) + \omega \hat{j} \times} (\cdot) = e^{\alpha \hat{j} \times} e^{\omega \hat{j} \times} = \begin{pmatrix} \cos \omega t & 0 & \sin \omega t \\ 0 & 1 & 0 \\ -\sin \omega t & 0 & \cos \omega t \end{pmatrix} \begin{pmatrix} e^{-\alpha t} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-\alpha t} \end{pmatrix} = \begin{pmatrix} e^{-\alpha t} \cos \omega t & 0 & e^{-\alpha t} \sin \omega t \\ 0 & 1 & 0 \\ -e^{-\alpha t} \sin \omega t & 0 & e^{-\alpha t} \cos \omega t \end{pmatrix}$$

for j -hat and

$$(2) e^{\alpha t(\hat{i} \times) + \alpha t(\hat{i} \times)^2}(\cdot) = e^{\alpha t(\hat{i} \times)} e^{\alpha t(\hat{i} \times)^2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \omega t & -\sin \omega t \\ 0 & \sin \omega t & \cos \omega t \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-\alpha t} & 0 \\ 0 & 0 & e^{-\alpha t} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-\alpha t} \cos \omega t & -e^{-\alpha t} \sin \omega t \\ 0 & e^{-\alpha t} \sin \omega t & e^{-\alpha t} \cos \omega t \end{pmatrix}$$

for i -hat. These the rotation matrices with damping exponentials, and are damped or converging (negative diverging) circulations. These expressions with both single and double crosses are therefore spirals. This has been without, yet, a single derivative or integral which is ultimately what this is all about. Hopefully this will be providing a basis for things to get very interesting. All things between Newton and everything. The theory of everything is calculus, that is, **diffusion calculus** in an **electromagnetic basis of heat** (i.e., an electromagnetic ether [basis] of heat; that is, a dot-matrix basis ether with all of the properties of thermodynamic heat [measured by T on the Kelvin scale] and heat flow Q).

Generalized Circles – the k -hat exponential with overtones

A Fourier transform of a signal for a frequency spectrum can be described as an excitation of a sample (or equivalent data) by a basis exponential (**expk**) and then a contour integration sequentially around the unit disk for a total winding defined by the resolution of the spectrometer. This is then scanned over all basis exponentials (all frequencies of **expk**) in the spectral/frequency (omega ω) region of interest. This is essentially the same thing as tuning with a basis exponential from an LC resonance for a radio-frequency signal. The peaks then show up at the residues (Cauchy integrals), that is, where there are basis exponentials in the signal. For tuning a radio, this is a peak for a k -hat exponential in the radio-frequency field {the basis} at the receiving antenna (which generates an oscillating current with that frequency at the resonance [k -hat exponential]). All waves require a basis to propagate, including electro-magnetic-radiation such as radio waves. This is also the excitation and resonance region of NMR (**Nuclear Magnetic Resonance**), that at which magnetic moments of nuclei resonate. Ultimately, spectroscopy is the purpose of this discussion of generalized circles in the context of the basis exponential **expk**. These are also visually appealing patterns when graphed as circles.

It was noted that the k -hat exponential is a circle on the real x - y plane once assigned a basis direction.

Here, the concept of basis exponential (here k -hat exponential) as a circle is extended to the concept of what I am terming generalised circles. I first started thinking about the concept of generalized circle with the power series document and how I was saying that power series is the expansion related to non-periodic functions (a finite power series fit to a harmonic function always results in a non-periodic approximation) and a Fourier series, obviously, relates to periodic functions. I stated that **power series** is a method of reconstructing a function from **successively adding unit roots** and a **Fourier series** is a method of reconstructing a (harmonic) function from **successively adding harmonic roots**. Harmonic roots are difficult to resolve, as compared to unit roots, into a large order series. The k -hat exponential is a simple method of transforming between the two. The purpose of this section is to introduce the concept of generalised circles from the ordinary standard circle, which is synonymous with the k -hat exponential, and place this concept into the context of spectroscopy. Generalized circles produce an infinite array of some very appealing geometry and also have some very important applications since they form cycles of arbitrary complexity. The most important of these applications is spectroscopy.

Cosines, being an even function, generate the dynamics on the x-axis (which is even as a power of $k \times$) and sines, being odd, generate the dynamics on the y-axis (which is odd as a power of $k \times$). These were hence previously known as the **real axis** and **imaginary axis**, now here referred to as the **even axis** and the **conjugate odd axis**.

The simplest way to generate a generalized circle is to take a polynomial function and replace the argument by the k -hat exponential. What results is a generalized circle. A **cosine series on the even axis** (x-axis) and the **conjugate sine series on the odd axis** (y-axis), where the conjugate sine series is defined as the series with the same coefficients as the even-cosine series but with the cosines replaced by odd-sines into the odd conjugate series. This is why I have named these generalised circles since it is an immediate generalization of the parametric form of a standard circle; cosine for even, sine for odd. Although essentially the same thing, a more technically sophisticated way is to expand the function in a power series by successively adding an infinite number of unit roots and then resolving these roots up to the k -hat exponential ($e^{\theta \hat{k} \times}$) rather than the forward time variable (t), that is, letting the forward time variable be a k -hat exponential (circular rather than linear time). This is a method of obtaining the Cauchy integral formula, which integrates the residue and amounts to the same thing as finding the Fourier coefficients, in this context, of a generalized circle. Note also that whenever integrating around a closed disk (the unit circle for example) in complex analysis, it is an integration of a k -hat, j -hat, or i -hat exponential (around the z , y , or x axes on the real R^2 x - y , x - z , or y - z planes).

A generalised circle with a polynomial of a single monomial term is de-Moivre's theorem. The power series expansion of a general polynomial with all terms ($n+1$ terms for a polynomial of order n) is then a method of using de-Moivre's theorem in a more general way, that is, to frequency scale the frequency components of a generalized circle rather than a single frequency of a standard circle.

Any parametric curve on a plane, or in a volume, that closes on itself then forms a circulation and is consequently harmonic and hence can be reconstructed as a generalized circle [although not Bowditch curves (?)] with a cosine series on the even axis and its sine conjugate series on the odd axis. Repetitions around the curve, synonymous with revolutions per minute on an ordinary circle, are known as windings in complex analysis, which are synonymous with frequency (of a cycle) when in the context of (per unit of) time.

Say for some function X , the **power series expansion** around a backward time variable (t_0) at the origin ($t_0 = 0$) and forward time variable (t) is

$$X(t) = x_0 + \frac{1}{1!}t D x_0 + \frac{1}{2!}t^2 D^2 x_0 + \frac{1}{3!}t^3 D^3 x_0 + \frac{1}{4!}t^4 D^4 x_0 + \frac{1}{5!}t^5 D^5 x_0 + \dots = \sum_{j=0}^{\infty} \frac{t}{j!} D^j x_0$$

where the forward time variable (t) is the upper limit of integration on the outer root. The integral itself only remains in the remainder term, which is not included since an infinite expansion is assumed. Polynomial powers of the forward time variable t weights the backward diffusion variables ($x_0, D x_0, D^2 x_0, \dots$) to form the polynomial expansion. For the Cauchy integral formula and the generalized circles, this function which is non-periodic, is converted to a periodic function by the outer integration (the actual integral isn't visible unless there is a remainder) by replacing the upper limit, the forward time variable, by the k -hat exponential ($t = e^{\theta \hat{k} \times}$)

$$X(e^{\theta \hat{k} \times}) = x_0 + \frac{1}{1!}(e^{\theta \hat{k} \times})^1 D x_0 + \frac{1}{2!}(e^{\theta \hat{k} \times})^2 D^2 x_0 + \frac{1}{3!}(e^{\theta \hat{k} \times})^3 D^3 x_0 + \dots + \frac{1}{(n-1)!}(e^{\theta \hat{k} \times})^{n-1} D^{n-1} x_0 + \frac{1}{n!}(e^{\theta \hat{k} \times})^n D^n x_0 + \frac{1}{(n+1)!}(e^{\theta \hat{k} \times})^{n+1} D^{n+1} x_0 + \dots$$

that is

$$X(e^{\theta \hat{k} \times}) = x_0 + \frac{1}{1!}e^{\theta \hat{k} \times} D x_0 + \frac{1}{2!}e^{2\theta \hat{k} \times} D^2 x_0 + \frac{1}{3!}e^{3\theta \hat{k} \times} D^3 x_0 + \dots + \frac{1}{(n-1)!}e^{(n-1)\theta \hat{k} \times} D^{n-1} x_0 + \frac{1}{n!}e^{n\theta \hat{k} \times} D^n x_0 + \frac{1}{(n+1)!}e^{(n+1)\theta \hat{k} \times} D^{n+1} x_0 + \dots$$

which is a generalized circle, where the backward diffusion (spatial) variables $x_0, Dx_0, D^2x_0, \dots$ are constants and are the initial conditions of the **infinite order differential equation generating the power series** [as demonstrated in my power series document]. From this expansion, the **Cauchy integral formula** will be briefly derived and then generalized circles discussed.

For the relevance of this expansion and noted at the outset, taking a **Fourier spectrum** is a **scanning search over frequency for k -hat exponentials**. These are closed contour integrals over each segment of frequency defined by the resolution of the search (resolution of the spectrometer for example). k -hat exponentials then show up as residues, which are integrated around a sufficient winding [to zero the non-residues] for a Cauchy integral formula of the residue [albeit ordinarily empirical with data]. Since each peak is a k -hat exponential at a particular frequency, any spectrum with two or more peaks is a generalized circle. For this reason, that a Fourier transform detects k -hat exponentials, the Fourier transform is arguably the **most important mathematical equation** (with that of the diffusion and heat equations) of **all of spectroscopy** [probably more so than the Gram stain is to bacteriology and microbiology in general]. The three concepts most relevant for this (scanning searches for k -hat exponentials by the Fourier transform, i.e., spectral analysis) is **Laurent series**, the **Cauchy Integral formula** for the **residue**, and the concept of **k -hat exponential** and **generalized circles** (polynomials of k -hat exponentials).

To detect the k -hat exponential at frequency n (which is the underlined term in the next line of maths) as a Fourier spectral peak, and consequently, for the **Cauchy integral formula** (with a further change of variable), dividing by the appropriate power of the k -hat exponential $e^{n\theta\hat{k}\times}$ to leave the residue of interest (this is the n^{th} excitation of the sample at frequency n for $\theta = \omega t$), $\frac{1}{n!}D^n x_0$, which extends the power series by analytical continuation into a **Laurent series** where the terms below the residue (to the left of the non-bracketed term) are the **principal part** and those above (to the right of the underlined term) are the **analytical part** (the **analytical part** being a series in the k -exponential and that of the **principal part** in the k -hat conjugate which is the **analytical continuation**)

$$e^{-n\theta\hat{k}\times} X(e^{\theta\hat{k}\times}) = e^{-n\theta\hat{k}\times} x_0 + \frac{1}{1!} e^{-(n-1)\theta\hat{k}\times} Dx_0 + \frac{1}{2!} e^{-(n-2)\theta\hat{k}\times} D^2x_0 + \dots + \frac{1}{(n-1)!} e^{-\theta\hat{k}\times} D^{n-1}x_0 + \frac{1}{n!} D^n x_0 + \frac{1}{(n+1)!} e^{\theta\hat{k}\times} D^{n+1}x_0 + \dots$$

$$= \left(e^{-n\theta\hat{k}\times} x_0 + \frac{1}{1!} e^{-(n-1)\theta\hat{k}\times} Dx_0 + \frac{1}{2!} e^{-(n-2)\theta\hat{k}\times} D^2x_0 + \dots + \frac{1}{(n-1)!} e^{-\theta\hat{k}\times} D^{n-1}x_0 \right) + \frac{1}{n!} D^n x_0 + \left(\frac{1}{(n+1)!} e^{\theta\hat{k}\times} D^{n+1}x_0 + \dots \right)$$

where $\frac{1}{n!} D^n x_0$

is the residue of $e^{-n\theta\hat{k}\times} X(e^{\theta\hat{k}\times})$

$$\frac{1}{(n+1)!} e^{\theta\hat{k}\times} D^{n+1}x_0 + \frac{1}{(n+2)!} e^{2\theta\hat{k}\times} D^{n+2}x_0 + \dots$$

is the analytical part of $e^{-n\theta\hat{k}\times} X(e^{\theta\hat{k}\times})$

$$e^{-n\theta\hat{k}\times} x_0 + \frac{1}{1!} e^{-(n-1)\theta\hat{k}\times} Dx_0 + \frac{1}{2!} e^{-(n-2)\theta\hat{k}\times} D^2x_0 + \dots + \frac{1}{(n-1)!} e^{-\theta\hat{k}\times} D^{n-1}x_0$$

is the principal part of $e^{-n\theta\hat{k}\times} X(e^{\theta\hat{k}\times})$

Integration around the contour of the k -hat exponential then zeroes all the terms in the expression except the residue, that is, zeroes all of the terms of the principal and analytical parts and leaves the residue which is independent of the k -hat exponential [after the k -hat excitation to a Laurent series] and the theta dependence of the contour integral

$$\oint e^{-n\theta\hat{k}\times} X(e^{\theta\hat{k}\times}) d\theta = \oint \left(e^{-n\theta\hat{k}\times} x_0 + \frac{1}{1!} e^{-(n-1)\theta\hat{k}\times} Dx_0 + \frac{1}{2!} e^{-(n-2)\theta\hat{k}\times} D^2x_0 + \dots + \frac{1}{(n-1)!} e^{-\theta\hat{k}\times} D^{n-1}x_0 + \frac{1}{n!} D^n x_0 + \frac{1}{(n+1)!} e^{\theta\hat{k}\times} D^{n+1}x_0 + \dots \right) d\theta$$

$$= \oint \left(e^{-n\theta\hat{k}\times} x_0 + \frac{1}{1!} e^{-(n-1)\theta\hat{k}\times} Dx_0 + \frac{1}{2!} e^{-(n-2)\theta\hat{k}\times} D^2x_0 + \dots + \frac{1}{(n-1)!} e^{-\theta\hat{k}\times} D^{n-1}x_0 \right) d\theta + \oint \left(\frac{1}{n!} D^n x_0 \right) d\theta + \oint \left(\frac{1}{(n+1)!} e^{\theta\hat{k}\times} D^{n+1}x_0 + \dots \right) d\theta$$

$$= \oint \left(\frac{1}{n!} D^n x_0 \right) d\theta$$

$$= \left(\frac{1}{n!} D^n x_0 \right) \oint d\theta$$

$$= 2\pi \left(\frac{1}{n!} D^n x_0 \right)$$

This is the Fourier spectral peak for the n^{th} $\hat{k}$ -hat exponential in the series (generalized circle) with frequency $n\omega_0$ ($\theta = \omega_0 t$) and the formula upon which all of spectral analysis is based and therefore the most important formula in this section, a form of the **Cauchy integral formula** for poles and zeroes of transfer functions and Fourier spectral data (that detects positions and magnitudes of poles)... we might be able to do this spatially for Coriolis and the like as well (for weather patterns for instance).

Further, for mathematical interests, changing back to the original forward time variable with the change of variable (in time t notation rather than complex z notation)

$$e^{\theta \hat{k} \times} = z - z_0 \quad dz = e^{\theta \hat{k} \times} (\hat{k} \times) d\theta \quad d\theta = \frac{dz}{e^{\theta \hat{k} \times} (\hat{k} \times)} = \frac{dz}{(z - z_0) \hat{k} \times} \quad (\text{standard complex theory with } z = t, z_0 = t_0, \text{ and } i = \hat{k} \times)$$

then results in the n^{th} order **Cauchy integral formula** for evaluating area integrals

$$\oint \frac{X(z)}{z^{n+1}} dz = 2\pi \left(\frac{1}{n!} D^n x_0 \right) \hat{k} \times \quad \oint \frac{X(z)}{(z - z_0)^{n+1}} dz = 2\pi \left(\frac{1}{n!} D^n x_0 \right) \hat{k} \times$$

depending if the residue (pole) is placed at the origin or at some offset z_0 from the origin (a backward time variable other than the origin). This is then repeated (frequency scanned) over all frequencies in the region of the spectrum of interest to obtain each of the coefficients. Note that the change of variable $e^{\theta \hat{k} \times} = z - z_0$ to the standard **Cauchy integral formula** is not actually required for spectroscopy since all that is required for spectroscopy is the location and magnitude of the peak. The formula for this, the coefficients of a Fourier spectrum is that of

$$\oint e^{-n\theta \hat{k} \times} X(e^{\theta \hat{k} \times}) d\theta = 2\pi \left(\frac{1}{n!} D^n x_0 \right) \quad n = 0, 1, 2, 3, \dots$$

Further changes of variable can be used to return the transfer function and the like which is not considered further in this document (other than a brief mention when discussing **basis factorisation of a matrix**). Also note that in most instances, this (Fourier spectral data acquisition) is implemented discretely, for instance, in discrete signal processing (it all works the same, though, with a high enough sampling frequency).

For **generalized circles**, by de-Moivre's theorem

$$\begin{aligned} f(e^{\theta \hat{k} \times}) &= f(0) + \frac{1}{1!} (\text{cosk } \theta + \text{sink } \theta \hat{k} \times) f'(0) + \frac{1}{2!} (\text{cosk } 2\theta + \text{sink } 2\theta \hat{k} \times) f''(0) + \frac{1}{3!} (\text{cosk } 3\theta + \text{sink } 3\theta \hat{k} \times) f'''(0) + \dots \\ &= f(0) + \frac{1}{1!} f'(0) \text{cosk } \theta + \frac{1}{2!} f''(0) \text{cosk } 2\theta + \frac{1}{3!} f'''(0) \text{cosk } 3\theta + \dots \\ &\quad + \frac{1}{1!} f'(0) \text{sink } \theta \hat{k} \times + \frac{1}{2!} f''(0) \text{sink } 2\theta \hat{k} \times + \frac{1}{3!} f'''(0) \text{sink } 3\theta \hat{k} \times + \dots \\ &= \left(f(0) + \frac{1}{1!} f'(0) \text{cosk } \theta + \frac{1}{2!} f''(0) \text{cosk } 2\theta + \frac{1}{3!} f'''(0) \text{cosk } 3\theta + \dots \right) \\ &\quad + \left(\frac{1}{1!} f'(0) \text{sink } \theta + \frac{1}{2!} f''(0) \text{sink } 2\theta + \frac{1}{3!} f'''(0) \text{sink } 3\theta + \dots \right) \hat{k} \times \\ &= \sum_{j=0}^{\infty} \left(\frac{1}{j!} f^{(j)}(0) \right) \text{cosk } j\theta + \sum_{j=0}^{\infty} \left(\frac{1}{j!} f^{(j)}(0) \right) \text{sink } j\theta \hat{k} \times \end{aligned}$$

which has the same form of a Fourier series expansion, except that **(1)** the functional dependence of X is the $\hat{k}$ -hat exponential rather than a linear variable and **(2)** the cosines and sines are **$\hat{k}$ -hat cosines (cosk)** and **conjugate $\hat{k}$ -hat sines (sink)**, and are hence operators, and **(3)** the odd terms in the $\hat{k}$ -hat sines are coefficient functions of $\hat{k} \times$. Until a basis direction is assigned, the expansion is an operator. The terms in the sine series would have previously been considered complex. Now they are here considered real on the

conjugate odd axis once a direction is assigned to X (or equivalently f). The cosines on the even axis (x -axis) and the sines on the odd axis (y -axis) if defined for i -hat (a zero phase basis). These are permuted if defined for j -hat. If you look up a table of Fourier expansions, 9 out of 10 of them are either odd or even expansions, that is, either expanded all in terms of cosines or all in terms of sines. Here it makes sense that this is a Fourier expansion for a curve on a plane where the **series of cosines reconstructs the even dynamics** and the **series of sines reconstructs the odd dynamics** (for the **even x axis** and **conjugate odd y axis**, respectively, in powers of $k \times$).

Example: Monomial powers for the function X is then statement of de-Moivre's theorem. A simple scale-invariant integrated constant acceleration is

$$X(t) = t^2$$

then $DX(t) = 2t$ and $D^2X(t) = 2$ is constant (so $x(0) = x_0 = 0$, then $Dx(0) = \dot{x}_0 = 0$ and $D^2x(0) = \ddot{x}_0 = 2$) and all other coefficients are zero so

$$X(e^{\theta \hat{k} \times}) = e^{2\theta \hat{k} \times} = \sum_{k=0}^{\infty} \left(\frac{1}{k!} D^k x_0 \right) \text{cosk } n\theta + \sum_{k=0}^{\infty} \left(\frac{1}{k!} D^k x_0 \right) \text{sink } n\theta \hat{k} \times = \left(\frac{1}{2!} D^2 x_0 \right) \text{cosk } 2\theta + \left(\frac{1}{2!} D^2 x_0 \right) \text{sink } 2\theta \hat{k} \times = \text{cosk } 2\theta + \text{sink } 2\theta \hat{k} \times$$

This was to demonstrate that a monomial power of a generalized circle is a frequency scaling of a standard circle k -hat exponential and hence an example of de-Moivre's theorem. This example is now put into the context of a force system and demonstrates a diadic expansion to the rotation matrix. Say that an arbitrary constant acceleration system is initially at the origin and with no initial velocity. This is equivalent to the monomial 2nd order power as the last example, but now with x -double-dot ($\ddot{x}_0$) as the initial constant acceleration. So the position vector x -component for the solved (integrated) force system is

$$X(t) = \frac{1}{2} \ddot{x}_0 t^2$$

then $DX(t) = \ddot{x}_0 t$ is the time t velocity and $D^2X(t) = \ddot{x}_0$ is the acceleration (which is the constant that parameterises the system). So $x(0) = x_0 = 0$, then $Dx(0) = \dot{x}_0 = 0$ and $D^2x(0) = \ddot{x}_0$) and all other coefficients are zero so

$$X(e^{\theta \hat{k} \times}) = \frac{1}{2} \ddot{x}_0 e^{2\theta \hat{k} \times} = \sum_{k=0}^{\infty} \left(\frac{1}{k!} D^k x_0 \right) \text{cosk } n\theta + \sum_{k=0}^{\infty} \left(\frac{1}{k!} D^k x_0 \right) \text{sink } n\theta \hat{k} \times = \left(\frac{1}{2!} D^2 x_0 \right) \text{cosk } 2\theta + \left(\frac{1}{2!} D^2 x_0 \right) \text{sink } 2\theta \hat{k} \times = \frac{1}{2} \ddot{x}_0 (\text{cosk } 2\theta + \text{sink } 2\theta \hat{k} \times)$$

which was for the X diffusion. If the force system is a constant acceleration system on each axis the Y diffusion and the Z diffusions are similar as the X diffusion but with accelerations $\ddot{y}_0$ and $\ddot{z}_0$

$$Y(t) = \frac{1}{2} \ddot{y}_0 t^2$$

and

$$Z(t) = \frac{1}{2} \ddot{z}_0 t^2$$

so

$$Y(e^{\theta \hat{k} \times}) = \frac{1}{2} \ddot{y}_0 e^{2\theta \hat{k} \times} = \sum_{k=0}^{\infty} \left(\frac{1}{k!} D^k y_0 \right) \text{cosk } n\theta + \sum_{k=0}^{\infty} \left(\frac{1}{k!} D^k y_0 \right) \text{sink } n\theta \hat{k} \times = \left(\frac{1}{2!} D^2 y_0 \right) \text{cosk } 2\theta + \left(\frac{1}{2!} D^2 y_0 \right) \text{sink } 2\theta \hat{k} \times = \frac{1}{2} \ddot{y}_0 (\text{cosk } 2\theta + \text{sink } 2\theta \hat{k} \times)$$

$$X(e^{\theta \hat{k} \times}) = \frac{1}{2} \ddot{z}_0 e^{2\theta \hat{k} \times} = \sum_{k=0}^{\infty} \left(\frac{1}{k!} D^k z_0 \right) \text{cosk } n\theta + \sum_{k=0}^{\infty} \left(\frac{1}{k!} D^k z_0 \right) \text{sink } n\theta \hat{k} \times = \left(\frac{1}{2!} D^2 z_0 \right) \text{cosk } 2\theta + \left(\frac{1}{2!} D^2 z_0 \right) \text{sink } 2\theta \hat{k} \times = \frac{1}{2} \ddot{z}_0 (\text{cosk } 2\theta + \text{sink } 2\theta \hat{k} \times)$$

and when defined in their own directions (i -hat for X , j -hat for Y , and k -hat for Z)

$$\begin{aligned}
X(e^{\theta \hat{k} \times}) \hat{i} &= \frac{1}{2} \ddot{x}_0 e^{2\theta \hat{k} \times} \hat{i} = \frac{1}{2} (\mathbf{cosk} \, 2\theta \hat{i} + \mathbf{sink} \, 2\theta \hat{k} \times \hat{i}) \ddot{x}_0 = \frac{1}{2} (\hat{i} \cos 2\theta + \hat{j} \sin 2\theta + \hat{k} \, 0) \ddot{x}_0 \\
Y(e^{\theta \hat{k} \times}) \hat{j} &= \frac{1}{2} \ddot{y}_0 e^{2\theta \hat{k} \times} \hat{j} = \frac{1}{2} (\mathbf{cosk} \, 2\theta \hat{j} + \mathbf{sink} \, 2\theta \hat{k} \times \hat{j}) \ddot{y}_0 = \frac{1}{2} (-\hat{i} \sin 2\theta + \hat{j} \cos 2\theta + \hat{k} \, 0) \ddot{y}_0 \\
Z(e^{\theta \hat{k} \times}) \hat{k} &= \frac{1}{2} \ddot{z}_0 e^{2\theta \hat{k} \times} \hat{k} = \frac{1}{2} (\mathbf{cosk} \, 2\theta \hat{k} + \mathbf{sink} \, 2\theta \hat{k} \times \hat{k}) \ddot{z}_0 = \frac{1}{2} (\hat{i} \, 0 + \hat{j} \, 0 + \hat{k} \, 1) \ddot{z}_0
\end{aligned}$$

diadically expanding with $\hat{i}'\hat{i} = 1$, $\hat{j}'\hat{j} = 1$, and $\hat{k}'\hat{k} = 1$

$$\begin{aligned}
X(e^{\theta \hat{k} \times}) \hat{i} &= \frac{1}{2} (\hat{i} \cos 2\theta + \hat{j} \sin 2\theta + \hat{k} \, 0) (\hat{i}'\hat{i}) \ddot{x}_0 = \frac{1}{2} (\hat{i}\hat{i}' \cos 2\theta + \hat{j}\hat{i}' \sin 2\theta + \hat{k}\hat{i}' \, 0) \ddot{x}_0 \hat{i} \\
Y(e^{\theta \hat{k} \times}) \hat{j} &= \frac{1}{2} (-\hat{i} \sin 2\theta + \hat{j} \cos 2\theta + \hat{k} \, 0) (\hat{j}'\hat{j}) \ddot{y}_0 = \frac{1}{2} (-\hat{i}\hat{j}' \sin 2\theta + \hat{j}\hat{j}' \cos 2\theta + \hat{k}\hat{j}' \, 0) \ddot{y}_0 \hat{j} \\
Z(e^{\theta \hat{k} \times}) \hat{k} &= \frac{1}{2} (\hat{i} \, 0 + \hat{j} \, 0 + \hat{k} \, 1) (\hat{k}'\hat{k}) \ddot{z}_0 = \frac{1}{2} (\hat{i}\hat{k}' \, 0 + \hat{j}\hat{k}' \, 0 + \hat{k}\hat{k}' \, 1) \ddot{z}_0 \hat{k}
\end{aligned}$$

that is,

$$\bar{R}(e^{\theta \hat{k} \times}) = \begin{pmatrix} X(e^{\theta \hat{k} \times}) \\ Y(e^{\theta \hat{k} \times}) \\ Z(e^{\theta \hat{k} \times}) \end{pmatrix} = \frac{1}{2} e^{2\theta \hat{k} \times} \begin{pmatrix} \ddot{x}_0 \\ \ddot{y}_0 \\ \ddot{z}_0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \cos 2\theta & -\sin 2\theta & 0 \\ \sin 2\theta & \cos 2\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \ddot{x}_0 \\ \ddot{y}_0 \\ \ddot{z}_0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \cos 2\theta & -\sin 2\theta & 0 \\ \sin 2\theta & \cos 2\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \vec{r}_0$$

which is the diadic expansion in detail. Wouldn't it be interesting if time itself was a k -hat exponential? although that's science fiction. What the concept of generalized circles is doing is assuming time is linear and then substituting it for a k -hat exponential (circles). Almost all, if not all in their entirety, of our measures of time use oscillators, that is, k -hat exponentials to measure and keep track of time, either physical oscillators or electronic oscillators or radiative oscillators. What I think is amazing is that there is all of this amazing dynamics obtained simply by substituting linear time with circles. Also of relevance here is that all solutions to Maxwells equations for electromagnetic light are k -hat exponentials.

As another simple example, let X be a quadratic function on the reals (rather than randomly variable, in which case it would be integrated Brownian motion)

$$X(t) = at^2 + bt + c$$

then $DX(t) = 2at + b$ and $D^2X(t) = 2a$ so $x(0) = x_0 = c$, $DX(0) = \dot{x}_0 = b$, and $D^2X(0) = \ddot{x}_0 = 2a$ so

$$\begin{aligned}
X(e^{\theta \hat{k} \times}) &= a(e^{\theta \hat{k} \times})^2 + b(e^{\theta \hat{k} \times})^1 + c(e^{\theta \hat{k} \times})^0 \\
&= a(\mathbf{cosk} \, 2\theta + \mathbf{sink} \, 2\theta \hat{k} \times) + b(\mathbf{cosk} \, 1\theta + \mathbf{sink} \, 1\theta \hat{k} \times) + c(\mathbf{cosk} \, 0\theta + \mathbf{sink} \, 0\theta \hat{k} \times) \\
&= (c + b \mathbf{cosk} \, \theta + a \mathbf{cosk} \, 2\theta) + (b \mathbf{sink} \, \theta + a \mathbf{sink} \, 2\theta) \hat{k} \times
\end{aligned}$$

which is the generalized circle composed of two harmonic frequencies, the fundamental and the first overtone, when applied to its basis direction i -hat. A diadic expansion is where an x -component is specified for the i -hat direction, a y -component is specified for the j -hat direction, and a z -component is specified for the k -hat direction. For the k -hat exponential, a single harmonic (a monomial power of a generalized circle), this didn't have to be explicit as such, any set of three scalars would do. For an arbitrary generalized circle, this does have to be explicit. This is interesting because it shows that it is both reasonable and necessary to have functions

specified solely for a particular direction, that is, for a particular basis-hat. This would be what an arbitrary plane curve is for a given k -hat exponential. Note that for an arbitrary plane curve, this could also be an amalgam of an X diffusion for the i -hat direction and a Y diffusion for the j -hat direction both contributing to the plane dynamics (via k -cross mapping to the other basis-hat) and maybe what has been a hidden facet to making further progress with standard mathematics [essentially caused a very large brick wall in tractability which may possibly now be overcome now it is clearer what is going on with the orthogonal axes and with what we knew as complex numbers]. Note also that j -hat and i -hat exponentials could also contribute to planar x - y dynamics via any of X , Y , or Z diffusion which could make things pretty complicated and difficult to sort out but at the same time provide for plenty of degrees of freedom to create sophisticated dynamics even when confined to a plane. Still hoping I will come across a general formula for Bowditch curves.

So for an arbitrary generalized circle, there is a diffusion variable X for the i -hat direction, a diffusion variable Y for the j -hat direction, and a diffusion variable Z for the k -hat direction. The diadic expansion proceeds similarly, but if there is anisotropy in X , Y , and or Z , then there are slightly more technical details for the expansion of the row and column vector bases to the matrix basis. I will keep with second order polynomial power series and put it in the context of a constantly accelerated system with the lower order terms of initial velocity and initial position other than the origin.

For a force-acceleration system that is started at some location (r_0) other than the origin and has some initial velocity ($\dot{r}_0$), the X diffusion has initial velocity $v_0 = \dot{x}_0$ and initial position x_0

$$X(t) = \frac{1}{2} \ddot{x}_0 t^2 + \dot{x}_0 t + x_0$$

and resolving with integration up to a k -hat exponential (circular rather than linear time)

$$\begin{aligned} X(e^{\theta \hat{k} \times}) &= \frac{1}{2} \ddot{x}_0 (e^{\theta \hat{k} \times})^2 + \dot{x}_0 (e^{\theta \hat{k} \times}) + x_0 \\ &= \frac{1}{2} \ddot{x}_0 (\text{cosk } 2\theta + \text{sink } 2\theta \hat{k} \times) + \dot{x}_0 (\text{cosk } \theta + \text{sink } \theta \hat{k} \times) + x_0 (\text{cosk } 0\theta + \text{sink } 0\theta \hat{k} \times) \\ &= (x_0 + \dot{x}_0 \text{cosk } \theta + \frac{1}{2} \ddot{x}_0 \text{cosk } 2\theta) + (\dot{x}_0 \text{sink } \theta + \frac{1}{2} \ddot{x}_0 \text{sink } 2\theta) \hat{k} \times \end{aligned}$$

which is the X diffusion defined for the i -hat basis variable, so

$$\begin{aligned} X(e^{\theta \hat{k} \times}) \hat{i} &= (x_0 + \dot{x}_0 \text{cosk } \theta + \frac{1}{2} \ddot{x}_0 \text{cosk } 2\theta) \hat{i} + (\dot{x}_0 \text{sink } \theta + \frac{1}{2} \ddot{x}_0 \text{sink } 2\theta) \hat{k} \times \hat{i} \\ &= (x_0 + \dot{x}_0 \text{cosk } \theta + \frac{1}{2} \ddot{x}_0 \text{cosk } 2\theta) \hat{i} + (\dot{x}_0 \text{sink } \theta + \frac{1}{2} \ddot{x}_0 \text{sink } 2\theta) \hat{j} \\ &= \hat{i} (x_0 + \dot{x}_0 \cos \theta + \frac{1}{2} \ddot{x}_0 \cos 2\theta) + \hat{j} (\dot{x}_0 \sin \theta + \frac{1}{2} \ddot{x}_0 \sin 2\theta) \end{aligned}$$

Now defining similarly the Y diffusion for the j -hat basis variable and the Z diffusion for the k -hat basis variable. These are

$$Y(t) = \frac{1}{2} \ddot{y}_0 t^2 + \dot{y}_0 t + y_0 \quad \text{and} \quad Z(t) = \frac{1}{2} \ddot{z}_0 t^2 + \dot{z}_0 t + z_0$$

and for the k -hat substitution for linear time

$$Y(e^{\theta \hat{k} \times}) = \frac{1}{2} \ddot{y}_0 (e^{\theta \hat{k} \times})^2 + \dot{y}_0 (e^{\theta \hat{k} \times}) + y_0 \quad \text{and} \quad Z(e^{\theta \hat{k} \times}) = \frac{1}{2} \ddot{z}_0 (e^{\theta \hat{k} \times})^2 + \dot{z}_0 (e^{\theta \hat{k} \times}) + z_0$$

and in their respective basis directions these are

$$\begin{aligned}
Y(e^{\theta \hat{k} \times}) \hat{j} &= (y_0 + \dot{y}_0 \text{cosk } \theta + \ddot{y}_0 \text{cosk } 2\theta) \hat{j} + (\dot{y}_0 \text{sink } \theta + \ddot{y}_0 \text{sink } 2\theta) \hat{k} \times \hat{j} \\
&= (y_0 + \dot{y}_0 \text{cosk } \theta + \ddot{y}_0 \text{cosk } 2\theta) \hat{j} - (\dot{y}_0 \text{sink } \theta + \ddot{y}_0 \text{sink } 2\theta) \hat{i} \\
&= \hat{j} (y_0 + \dot{y}_0 \cos \theta + \ddot{y}_0 \cos 2\theta) - \hat{i} (\dot{y}_0 \sin \theta + \ddot{y}_0 \sin 2\theta)
\end{aligned}$$

and in the k -hat direction, the direction of the k -hat exponential generating the generalized circle, which degenerates to a scaling of k -hat (the eigenvector of the generalized circle spin axis)

$$\begin{aligned}
Z(e^{\theta \hat{k} \times}) \hat{k} &= (z_0 + \dot{z}_0 \text{cosk } \theta + \ddot{z}_0 \text{cosk } 2\theta) \hat{k} + (\dot{z}_0 \text{sink } \theta + \ddot{z}_0 \text{sink } 2\theta) \cancel{\hat{k} \times \hat{k}} \\
&= (z_0 + \dot{z}_0 \text{cosk } \theta + \ddot{z}_0 \text{cosk } 2\theta) \hat{k} \\
&= \hat{k} (z_0 + \dot{z}_0 + \ddot{z}_0)
\end{aligned}$$

which can be normalized for a unity eigenvalue for $\hat{k}$ by dividing the position vector through by $z_0 + \dot{z}_0 + \ddot{z}_0$

In summary, the components are

$$\begin{aligned}
X(e^{\theta \hat{k} \times}) \hat{i} &= \hat{i} (x_0 + \dot{x}_0 \cos \theta + \tfrac{1}{2} \ddot{x}_0 \cos 2\theta) + \hat{j} (\dot{x}_0 \sin \theta + \tfrac{1}{2} \ddot{x}_0 \sin 2\theta) \\
Y(e^{\theta \hat{k} \times}) \hat{j} &= \hat{j} (y_0 + \dot{y}_0 \cos \theta + \ddot{y}_0 \cos 2\theta) - \hat{i} (\dot{y}_0 \sin \theta + \ddot{y}_0 \sin 2\theta) \\
Z(e^{\theta \hat{k} \times}) \hat{k} &= \hat{k} (z_0 + \dot{z}_0 + \ddot{z}_0)
\end{aligned}$$

Diadically expanding by putting $1 = \hat{i} \hat{i}$, $1 = \hat{j} \hat{j}$, and $1 = \hat{k} \hat{k}$ in the appropriate places (placing $1 = \hat{i} \hat{i}$, $1 = \hat{j} \hat{j}$, and $1 = \hat{k} \hat{k}$ on each of the positions, velocities and accelerations) and expanding and regrouping the vector and column basis variables into a matrix basis times a column basis maps the components to columns

$$\begin{pmatrix} X(e^{\theta \hat{k} \times}) \\ Y(e^{\theta \hat{k} \times}) \\ Z(e^{\theta \hat{k} \times}) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \dot{x}_0 \\ \dot{y}_0 \\ \dot{z}_0 \end{pmatrix} + \tfrac{1}{2} \begin{pmatrix} \cos 2\theta & -\sin 2\theta & 0 \\ \sin 2\theta & \cos 2\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \ddot{x}_0 \\ \ddot{y}_0 \\ \ddot{z}_0 \end{pmatrix}$$

which once you know how it all works one could just go like this

$$\begin{pmatrix} X(e^{\theta \hat{k} \times}) \\ Y(e^{\theta \hat{k} \times}) \\ Z(e^{\theta \hat{k} \times}) \end{pmatrix} = \begin{pmatrix} \tfrac{1}{2} \ddot{x}_0 (e^{\theta \hat{k} \times})^2 + \dot{x}_0 (e^{\theta \hat{k} \times}) + x_0 \\ \tfrac{1}{2} \ddot{y}_0 (e^{\theta \hat{k} \times})^2 + \dot{y}_0 (e^{\theta \hat{k} \times}) + y_0 \\ \tfrac{1}{2} \ddot{z}_0 (e^{\theta \hat{k} \times})^2 + \dot{z}_0 (e^{\theta \hat{k} \times}) + z_0 \end{pmatrix} = \tfrac{1}{2} (e^{\theta \hat{k} \times})^2 \begin{pmatrix} \ddot{x}_0 \\ \ddot{y}_0 \\ \ddot{z}_0 \end{pmatrix} + (e^{\theta \hat{k} \times})^1 \begin{pmatrix} \dot{x}_0 \\ \dot{y}_0 \\ \dot{z}_0 \end{pmatrix} + (e^{\theta \hat{k} \times})^0 \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$$

and then put the k -hat exponentials into rotation matrix form which will result in the previous diadic expansion. This last form in terms of the k -hat exponentials has the form of the polynomial function $\bar{R}(t) = r_0 + \dot{r}_0 t^1 + \ddot{r}_0 t^2$ with $t = \exp(\theta \hat{k} \times)$ which is that expressed in terms of eigenvectors and eigenvalues and would be that on which to apply a Fourier transform for a frequency spectrum, in which case, to detect here the two frequencies of the velocity fundamental and the acceleration overtone. Note that this, again, has been without a single bit of calculus for dynamics. When I add to this the calculus, I am suspecting there is going to be some very spectacular theory (which I will be demonstrating with simulations).

The components can also be centred and scaled, as per standardisation for statistical theory for a Gauss basis

$$\begin{aligned} \frac{X(e^{\theta \hat{k} \times}) - \dot{x}_0}{\dot{x}_0} \hat{i} &= \hat{i} (\cos \theta + (\ddot{x}_0 / \dot{x}_0) \cos 2\theta) + \hat{j} (\sin \theta + (\ddot{x}_0 / \dot{x}_0) \sin 2\theta) \\ \frac{Y(e^{\theta \hat{k} \times}) - \dot{y}_0}{\dot{y}_0} \hat{j} &= \hat{j} (\cos \theta + (\ddot{y}_0 / \dot{y}_0) \cos 2\theta) - \hat{i} (\sin \theta + (\ddot{y}_0 / \dot{y}_0) \sin 2\theta) \\ \frac{Z(e^{\theta \hat{k} \times}) - \dot{z}_0}{\dot{z}_0} \hat{k} &= \hat{k} (1 + \ddot{z}_0 / \dot{z}_0) \end{aligned}$$

and would be two concentric tori of Gauss density (for a Gauss distributed basis).

Here $\theta = \omega_0 t$ to keep the maths uncluttered. The fundamental frequency (the base frequency) ω_0 is the most important parameter of a k -hat exponential and a generalized circles of k -hat exponential powers. The Fourier transform of $X(e^{\theta \hat{k} \times}) = c + be^{\theta \hat{k} \times} + ae^{2\theta \hat{k} \times}$ is then the Fourier frequency spectrum for the generalized circle which returns Dirac deltas at the respective frequencies (de-Moivre powers of the k -hat exponential) weighted by the coefficients

$$\phi(\omega) = b \delta(\omega - 1) + a \delta(\omega - 2)$$

Because the Dirac pulse at a frequency is [quasi-]infinite [since the resolution of the spectrometer is quasi-infinitesimal], the integral can be slightly redefined, as an integral average. If the step length of the scanning search of the spectrometer is $d\omega = h$ (that is, where ω is scanned in steps/increments of length $d\omega = h$), then to match this in the integral limit (for a total winding to zero all the non-residues) the integrand could be scaled down by the same limit (the total winding $= 2\pi / d\omega = 2\pi / h$). This would be a mean (of the integral) and would leave the residue at the frequency of the k -hat exponential equal to the coefficient of that term (rather than being [quasi-]infinite [reciprocal $d\omega = h$]).

$$\frac{1}{n!} D^n x_0 = (2\pi / h)^{-1} \int_0^{2\pi/h} e^{-n\theta \hat{k} \times} X(e^{\theta \hat{k} \times}) d\theta \quad \text{i.e.,} \quad \begin{aligned} \theta &= \omega_0 t \\ d\theta &= \omega_0 dt \end{aligned} \quad \frac{1}{n!} D^n x_0 = (\omega_0 h / 2\pi) \int_0^{2\pi/h} e^{-n\theta \hat{k} \times} X(e^{\theta \hat{k} \times}) d\theta$$

which is a Fourier coefficient and Cauchy integral. Note that the total winding could also be an integral from $-\pi / h$ to $+\pi / h$. So rescaling by the resolution of the frequency scan then normalizes the coefficients and converts the deltas to unit Dirac deltas where here $\delta(\omega) = 1$ when $\omega = 0$ and $\delta(\omega) = 0$ otherwise ($\omega \neq 0$). The coefficients are powers of the respective differential order when the polynomial is a power series expansion of the generating function. If a resonant frequency of ω_0 is scaled onto the k -hat exponential (by de-Moivre's theorem) for $T = e^{\omega_0 \theta \hat{k} \times}$ and consequently $X(e^{\omega_0 \theta \hat{k} \times})$ which is then the fundamental frequency with one overtone, then the spectrum would display

$$\phi(\omega) = b \delta(\omega - \omega_0) + a \delta(\omega - 2\omega_0)$$

graphed as ϕ as a function of ω . At a specific frequency, the transform converts the signal into a Laurent series where all terms of $X(e^{\omega_0 \theta \hat{k} \times})$ below that at that frequency (ω_0) are principal terms and all terms above are analytic terms. This leaves the term itself as the residue. The integration is a contour integral around the unit disk with winding length $2\pi/h$ (where h = spectral resolution of the ω frequency scan) to zero the principal and analytical terms to leave the residue as a single integral. The integration limit (of 2π divided

by the resolution of the spectrometer) is because one step outside the resonant frequency present in the frequency scan is the smallest (net) frequency ($\omega - \omega_0 = h$) and hence the largest wavelength residual in the scan and so the smallest total winding (for a winding number of one for this term [one residual wavelength of $\omega - \omega_0 = h$], integration from 0 to $2\pi/h$) that will zero all non-residue terms with a closed contour integration. That is, a Fourier transform for a spectrum integrates as a contour over a total winding sufficient to zero all non-residues in the Laurent expansion (analytical continuation of a power series expansion after excitation by a k -hat exponential) for a particular frequency. The k -hat exponentials just outside excitation determines the largest total integration (for a total winding number of 1 wavelength) to zero those terms (a total winding number of 1 for these two terms on either side of the excitation $\omega - \omega_0 = \pm h$), a total winding number of 2 (two total wavelengths) for two steps just outside the excitation $\omega - \omega_0 = \pm 2h$ and so on. I don't know for sure, but this is probably how the Fast Fourier Transform improves efficiency, by reducing the integration region for those terms further away from the [excitation and hence] residue and hence requiring smaller integration regions to zero them [which would otherwise have very large winding numbers from very high residual frequencies, and hence small residual wavelengths requiring only a small integration region to zero $\Rightarrow$ I know what it is now, and it is not this, although integral winding numbers are still relevant though].

Note further that a generalized circle expansion of the k -hat exponential for a spectrum can also be expanded into splittings with a coupling constant J , for **NMR** for example (known as **MRI**, **Medical Resonance Imaging**, in the medical field)[the machines for which are **true marvels of medical science** (my mother used to take x-rays and MRIs for a living so I have spent much time in hospitals behind the controls watching these take place)]. Such a splitting, for an NMR, is essentially the same as that observed in Zeeman splitting of spectral lines in the presence of a magnetic field, as well as the equivalent with the Stark effect for the electric field. These are binomial powers (binomial splittings) of **cosk**. These would essentially then be integrated systems of a large number of interacting forces, similarly as almost all engineered systems. [note that my experience with NMR (rather than MRI) has been with first and second year organic chemistry CHEM1231 and CHEM2021 and the structure determination compilation by Field, Sternhell, and Kalman]

So then the question that may be asked is, so how is the polynomial chosen from which a generalized circle can be formed? The polynomial is the power series expansion of any function of interest or relevance. The powers at each frequency are then the unit roots that reconstruct that function as a series of differential dynamics of ascending order of sophistication and differential powers. It is, as was just demonstrated, also that which is determined empirically by spectroscopy in a frequency spectrum which then displays the coefficients of this expansion.

These circles can fit many [I would like to say all] functions on the plane [but not Bowditch curves] that close on themselves, and are hence harmonic (since they repeat). This represents a tremendous variety of curves (infinite variety actually) and models any closed cycle [but not Bowditch curves]. Even the simplest functions in this form are very appealing visually. A simple form of a generalized circle, and those I am most pleased about (having recently bought myself a spirograph [from BigW]), are **epicycloids** and **hypocycloids**, and **epitrochoids** and **hypotrochoids**. Hypotrochoids are **spirographs** (the deluxe set also has epitrochoids). These are circles rolling around inside another circle (or outside for epicycloids and epitrochoids) and are hence **circles within circles**. Hence there are two cosines on the even axis and two sines on conjugate odd axis. Naturally, the mathematics easily extends to **circles within circles within circles**. There are three cosines on the even axis and three sines on the conjugate odd axis. This can be taken to any order and is how power is transmitted through mechanical systems of crankshafts and gears since spinning shafts transmit power to other spinning shafts by way of circles (the gears) rolling on other circles with no-slip (cogs).

Note also interestingly, hypotrochoids (spirographs) also include ellipses (as I saw when playing with my spirograph) which are the **equations of motion of planetary orbits** [Kepler's first law]. These are hypotrochoids where the radius (R) of the outer circle is twice the radius (r) of the small inner rolling circle [with $h < r$]. These are the most simple hypotrochoids [above oscillating lines (the oscillators)]. Just a small increase in complexity (a small variation of the rolling parameters) would easily model the advance of a perihelion, such as that of Mercury. With an accurate mathematical equation for the **advance of perihelion** in the **orbital equation of motion**, a physical explanation could be put forward as to the physical source of the **cycloidal overtone** without the need for nonsense non-Newton theory [are satellites put up into orbit with nonsense theory? No they aren't, they are put up with Newton's Law for gravitation (to calculate orbits). Are mega skyscrapers made possible with nonsense? no, Newton's laws makes these structures possible (by calculating forces)]. Supposedly the fact that orbits are ellipses, that of Kepler's first law, broke with the premise from the ancient Athenians that everything was systems of perfect circles. With a spirograph, it is clear that an ellipse is a system of two interacting perfect circles so, after all, the Athenian premise {Euclid and Pythagoras for example} still holds true. An ellipse is a system of the interaction of two perfect circles. It is also a system of a $\hat{k}$ exponential interacting with its conjugate, both of which are perfect circles. The Fourier transform and its all important use in spectroscopy also re-affirms this premise of perfect circles.

For reference (Lawrence J. D., *A catalogue of special plane curves* and Lipschutz S., *Differential geometry*) [familiar to me after a 3rd year course in differential geometry (John Steele MATH3521)] the equations of epicycloids, epitrochoids, hypocycloids, and hypotrochoids are

R = radius of fixed circle

r = radius of rolling circle

h = radius from centre of rolling circle of radius r

epicycloid, circle rolling outside another circle

$$\vec{r} = (R + r)\mathbf{expk}\theta + r\mathbf{expk}(\theta(R + r)/r)$$

with components

$$x = (R + r)\cos\theta + r\cos\left(\frac{R + r}{r}\theta\right)$$

$$y = (R + r)\sin\theta + r\sin\left(\frac{R + r}{r}\theta\right)$$

hypocycloid, circle rolling inside another circle (epicycloid with $r = -r$)

$$\vec{r} = (R - r)\mathbf{expk}\theta - r\mathbf{expk}(-\theta(R - r)/r)$$

with components

$$x = (R + (-r))\cos\theta + (-r)\cos\left(\frac{R + (-r)}{(-r)}\theta\right)$$

$$y = (R + (-r))\sin\theta + (-r)\sin\left(\frac{R + (-r)}{(-r)}\theta\right)$$

epitrochoid

$$\vec{r} = (R + r)\mathbf{expk}\theta + h\mathbf{expk}(\theta(R + r)/r)$$

$$x = (R + r)\cos\theta + h\cos\left(\frac{R + r}{r}\theta\right)$$

$$y = (R + r)\sin\theta + h\sin\left(\frac{R + r}{r}\theta\right)$$

hypertrochoid (epitrochoid with $r = -r$ and $h = -h$)

$$\vec{r} = (R - r)\mathbf{expk}\theta - h\mathbf{expk}(-\theta(R - r)/r)$$

$$x = (R + (-r))\cos\theta + (-h)\cos\left(\frac{R + (-r)}{(-r)}\theta\right)$$

$$y = (R + (-r))\sin\theta + (-h)\sin\left(\frac{R + (-r)}{(-r)}\theta\right)$$

any of which would show up with a Fourier transform for a frequency spectrum on the x -data and on the y -data as two peaks similar as the example, the fundamental at $\theta = \omega_0 t$ and the overtone at $\theta_1 = \omega_0 t (R + r)/r$ for the outside (epi) rolling circle or $\omega_1 = (R - r)/r$ for the inside (hypo) rolling circle.

Generalized circles are a Fourier's theorem for expansions of periodic scalar functions to curves in the plane defined by power series polynomials of **expk**. The generalized circle comprises a harmonic **cosine series expansion as the even x-component**, and the **conjugate sine series expansion as the odd y component**. The Fourier coefficients are obtained in the standard way which corresponds to excitation by k -hat exponentials and evaluated by the Cauchy integral formula of the generalized circle converted to a Laurent series by the excitation.

This has resulted from defining the plane, as a redefinition of the Argand diagram on the reals, as the quartic roots ($k \times$) of the unit scalar eigenvector. These extend further to differential operators and then further again to a Gauss distributed basis which are hence determined empirically with spectroscopy. I will be doing this in later documents. I had originally thought this document was too trivial and was going to just use $k \times$ in these higher order documents that I just noted that I will be writing. I think it has been demonstrated that this additional document has been necessary to cover the theory, that of $k \times$, in detail [I am quite pleased about all of this spectroscopy theory as well, as the infinite limits on Fourier integrals had been another very vexing issue in the courses I have been taking, most notably, Circuits and Signals, Julian Epps ELEC2521, and the course textbook by Charles Alexander].

Diadically Expanded Matrix of a Generalized Circle

For the conversion of a generalized circle for an arbitrary polynomial function X expanded as a power series

$$\begin{aligned} \begin{pmatrix} X(e^{\theta \hat{k} \times}) \\ Y(e^{\theta \hat{k} \times}) \\ Z(e^{\theta \hat{k} \times}) \end{pmatrix} &= \sum_{n=0}^{\infty} e^{n\theta \hat{k} \times} \begin{pmatrix} \frac{1}{n!} D^n x_0 \\ \frac{1}{n!} D^n y_0 \\ \frac{1}{n!} D^n z_0 \end{pmatrix} = \sum_{n=0}^{\infty} R_z(n\theta) \begin{pmatrix} \frac{1}{n!} D^n x_0 \\ \frac{1}{n!} D^n y_0 \\ \frac{1}{n!} D^n z_0 \end{pmatrix} = \sum_{n=0}^{\infty} \begin{pmatrix} \cos n\theta & -\sin n\theta & 0 \\ \sin n\theta & \cos n\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{n!} D^n x_0 \\ \frac{1}{n!} D^n y_0 \\ \frac{1}{n!} D^n z_0 \end{pmatrix} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} e^{n\theta \hat{k} \times} \begin{pmatrix} D^n x_0 \\ D^n y_0 \\ D^n z_0 \end{pmatrix} = \sum_{n=0}^{\infty} \frac{1}{n!} \begin{pmatrix} \cos n\theta & -\sin n\theta & 0 \\ \sin n\theta & \cos n\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} D^n x_0 \\ D^n y_0 \\ D^n z_0 \end{pmatrix} \end{aligned}$$

This can be damped/attenuated with a double cross basis-hat with the same unitary eigenvector (the k -hat double cross in this instance). This would correspond to NMR relaxation time if the attenuation constant is uniform over all of the k -hat exponentials in the spectrum. This may not be unreasonable given that they are probably all chained (and probably attenuate like a mechanical clock when the power source runs down).

This description of the Fourier transform discussed in this topic of generalized circles, that of a contour integration of a Laurent series (after conversion from a power series with a k -hat excitation), accounts for the very vexing issue of infinite limits on the integrals for which sines and cosines do not converge and hence do not indicate valid integrals. I think that the Fourier transform for a spectrum described as 'a search for k -hat exponentials' (by analytical continuation and hence conversions to Laurent series) has provided an entirely satisfactory description of the important technique of Fourier spectral analysis (especially for spectroscopy). It also agrees with the common conception, among some of the best mathematical thinkers (Ian Stewart for example), that everything is circles within circles (in the context of calculus).

As a final word for this section, a clock is a k -hat exponential, and the mechanism of mechanical clocks is that of chained k -hat exponentials, that is, a generalized circle of k -hat exponentials which transmit the power and time-keeping-frequency (in a clockwork Universe) from the pendulum bob or pendulum spring. It is the pendulum bob or pendulum spring which would be the main attenuation factor for the power flow (with a small amount of friction), which would be quite reasonably exponential attenuation (similarly as batteries and Coulomb charge dissipation). This is especially interesting in the context of the human preoccupation with keeping time and is also interesting because time is the parameter of dynamics. It is also interesting because the most accurate measure of time now, as per the SI definition for unit of time, is from Caesium spectral transitions [SI Chemical Data and CRC Handbook of Chemistry and Physics]. The physics of these atomic transitions, and the decay chains, are probably mathematically the same as a mechanical clock (although with the extension from R to S [a Gauss basis]), that is, systems of chained k -hat exponentials. Aaron's Lemma [I named it after myself] can convert continuous half-life data to Poisson rate-intensity counts [this will be the next big research item in nuclear science, despite myself being a nuclear conservative because of the life-span reduction and anti-eugenic effects on human populations and what is happening to Earth's water].

Generalized circle: Cauchy-Riemann equations and Laplace's equation

This section is to show that generalized circles are harmonic by the definition from complex theory of the **Cauchy-Riemann equations** and **Laplace's equation**.

From complex theory (E.G Phillips, *functions of a complex variable* and David Angel, *Complex Analysis* MATH2521), a function of a complex variable [real basis curl] is said to be **analytic** [differentiable on the complex plane (R^2 vector plane)] when that function satisfies the **Cauchy-Riemann equations** and **harmonic** when it also satisfies **Laplace's equation**. If $z = x + iy$ is the complex variable and $f(z) = u + iv$ is the function of the complex variable ($z(x, y) = x + iy$ and $f(z) = u(x, y) + iv(x, y)$) then f is **analytic** (differentiable/holomorphic) if $u_x = v_y$ and $u_y = -v_x$ (Cauchy-Riemann equations) and **harmonic** if $u_{xx} + u_{yy} = 0$ and $v_{xx} + v_{yy} = 0$ (Laplace's equation for u and v). For a generalized circle, the function z is the k -hat exponential **expk** and x and y are the cosine and sine of the k -hat exponential with the fundamental frequency of the base k -hat exponential (here for $\theta = \omega_0 t$ for an arbitrary ω_0), where the cosine and sine are k -hat cosines **cosk** and k -hat sines **sink** until directions are assigned with basis-hats and so then revert to scalars, i.e., scalar cosine and sine after operating on the standard basis.

The function f is then the polynomial function that converts the k -hat exponential (which is a standard circle) into the generalized circle. This polynomial is any polynomial but ultimately a power series expansion of derivatives at the backward diffusion [spatial] variable. Here, either the notation X is used since it is a diffusion variable for a spatial coordinate (which also maps to Y since it is a function of k -cross) – or the functional f notation (functional rather than diffusion/spatial notation). For this section, the functional notation f is primarily adopted but can be changed over at will. The **even cosine expansion** (of the generalized circle) is the function $u = u(x, y)$ and the **conjugate odd sine expansion** is the function $v = v(x, y)$.

The **complex variable** is $z(x, y) = x + iy$ which, in standard complex notation, is the complex exponential

$$z(x, y) = x + iy = e^{i\theta} = \cos\theta + i\sin\theta$$

for the parameterization $x = \cos\theta$ and $y = \sin\theta$. In the complete notation this is the k -hat exponential

$$z(x, y) = x + iy = e^{\theta \hat{k} \times} = \text{cosk} \theta + (\text{sink} \theta) \hat{k} \times$$

which is equivalent (to the complex exponential/rotation matrix) in the i -hat and j -hat directions once one of these basis directions are assigned (and i is understood to be k -cross and the imaginary axis is the axis generated by j -hat, that is, the standard y -axis). Once a direction is assigned (i.e., when the basis functions operate on the standard basis), the x and y components of the base circle are those that parameterize the complex variable (real R^2 vector variable)

$$x = \cos \theta \quad y = \sin \theta$$

The **generalized circle** is then $f(z) = u + iv = u(x, y) + i v(x, y)$ which, in standard complex and $i = \hat{k} \times$ notation, is

$$f(e^{i\theta}) = f(0) + \frac{1}{1!} e^{i\theta} f'(0) + \frac{1}{2!} e^{2i\theta} f''(0) + \frac{1}{3!} e^{3i\theta} f'''(0) + \dots + \frac{1}{(n-1)!} e^{(n-1)i\theta} f^{(n-1)}(0) + \frac{1}{n!} e^{ni\theta} f^{(n)}(0) + \frac{1}{(n+1)!} e^{(n+1)i\theta} f^{(n+1)}(0) + \dots$$

$$f(e^{\theta \hat{k} \times}) = f(0) + \frac{1}{1!} e^{\theta \hat{k} \times} f'(0) + \frac{1}{2!} e^{2\theta \hat{k} \times} f''(0) + \frac{1}{3!} e^{3\theta \hat{k} \times} f'''(0) + \dots + \frac{1}{(n-1)!} e^{(n-1)\theta \hat{k} \times} f^{(n-1)}(0) + \frac{1}{n!} e^{n\theta \hat{k} \times} f^{(n)}(0) + \frac{1}{(n+1)!} e^{(n+1)\theta \hat{k} \times} f^{(n+1)}(0) + \dots$$

which is a power series expansion of **expk** which explicitly shows the differential dynamics at the backward variables. Expanding the k -hat exponential **expk** at the harmonic frequencies with the Euler-formula

$$\begin{aligned} f(e^{\theta \hat{k} \times}) &= f(0) + \frac{1}{1!} (\text{cosk} \theta + \text{sink} \theta \hat{k} \times) f'(0) + \frac{1}{2!} (\text{cosk} 2\theta + \text{sink} 2\theta \hat{k} \times) f''(0) + \frac{1}{3!} (\text{cosk} 3\theta + \text{sink} 3\theta \hat{k} \times) f'''(0) + \dots \\ &= f(0) + \frac{1}{1!} f'(0) \text{cosk} \theta + \frac{1}{2!} f''(0) \text{cosk} 2\theta + \frac{1}{3!} f'''(0) \text{cosk} 3\theta + \dots \\ &\quad + \frac{1}{1!} f'(0) \text{sink} \theta \hat{k} \times + \frac{1}{2!} f''(0) \text{sink} 2\theta \hat{k} \times + \frac{1}{3!} f'''(0) \text{sink} 3\theta \hat{k} \times + \dots \\ &= \left(f(0) + \frac{1}{1!} f'(0) \text{cosk} \theta + \frac{1}{2!} f''(0) \text{cosk} 2\theta + \frac{1}{3!} f'''(0) \text{cosk} 3\theta + \dots \right) \\ &\quad + \left(\frac{1}{1!} f'(0) \text{sink} \theta + \frac{1}{2!} f''(0) \text{sink} 2\theta + \frac{1}{3!} f'''(0) \text{sink} 3\theta + \dots \right) \hat{k} \times \\ &= \sum_{j=0}^{\infty} \left(\frac{1}{j!} f^{(j)}(0) \right) \text{cosk} j\theta + \sum_{j=0}^{\infty} \left(\frac{1}{j!} f^{(j)}(0) \right) \text{sink} j\theta \hat{k} \times \end{aligned}$$

Given that complex notation is more concise, despite not as theoretically complete for the R^3 vector space since i can arbitrarily commute whilst k -cross can't for example, in complex notation this is

$$\begin{aligned} f(e^{\theta \hat{k} \times}) &= \left(f(0) + \frac{1}{1!} f'(0) \cos \theta + \frac{1}{2!} f''(0) \cos 2\theta + \frac{1}{3!} f'''(0) \cos 3\theta + \dots \right) \\ &\quad + \left(\frac{1}{1!} f'(0) \sin \theta + \frac{1}{2!} f''(0) \sin 2\theta + \frac{1}{3!} f'''(0) \sin 3\theta + \dots \right) \hat{k} \times \\ &= \sum_{j=0}^{\infty} \left(\frac{1}{j!} f^{(j)}(0) \right) \cos j\theta + i \sum_{j=0}^{\infty} \left(\frac{1}{j!} f^{(j)}(0) \right) \sin j\theta \end{aligned}$$

The even and odd components of the generalized circle are then the functions u and v which are the cosine and sine expansions

$$u(x, y) = u(\cos \theta, \sin \theta) = \sum_{j=0}^{\infty} \left(\frac{1}{j!} f^{(j)}(0) \right) \cos j\theta \quad v(x, y) = v(\cos \theta, \sin \theta) = \sum_{j=0}^{\infty} \left(\frac{1}{j!} f^{(j)}(0) \right) \sin j\theta$$

which is a Fourier cosine expansion for the even component u and the conjugate Fourier sine expansion for the conjugate odd component v . Expressing these expansions firstly in terms of **expk** and then to the components of **expk** so that they can be differentiated for the Cauchy-Reimann and Laplace equations

$$\begin{aligned}
u(x, y) &= u(\cos \theta, \sin \theta) = \sum_{j=0}^{\infty} \left(\frac{1}{j!} f^{(j)}(0) \right) \cos j\theta = \sum_{j=0}^{\infty} \left(\frac{1}{j!} f^{(j)}(0) \right) \frac{e^{i(j\theta)} + e^{-i(j\theta)}}{2} \\
&= \sum_{j=0}^{\infty} \left(\frac{1}{j!} f^{(j)}(0) \right) \frac{(\cos \theta + i \sin \theta)^j + (\cos \theta - i \sin \theta)^j}{2} \\
&= \sum_{j=0}^{\infty} \left(\frac{1}{j!} f^{(j)}(0) \right) \frac{(x + i y)^j + (x - i y)^j}{2}
\end{aligned}$$

$$\begin{aligned}
v(x, y) &= u(\cos \theta, \sin \theta) = \sum_{j=0}^{\infty} \left(\frac{1}{j!} f^{(j)}(0) \right) \sin j\theta = \sum_{j=0}^{\infty} \left(\frac{1}{j!} f^{(j)}(0) \right) \frac{e^{i(j\theta)} - e^{-i(j\theta)}}{2i} \\
&= \sum_{j=0}^{\infty} \left(\frac{1}{j!} f^{(j)}(0) \right) \frac{(\cos \theta + i \sin \theta)^j - (\cos \theta - i \sin \theta)^j}{2i} \\
&= \sum_{j=0}^{\infty} \left(\frac{1}{j!} f^{(j)}(0) \right) \frac{(x + i y)^j - (x - i y)^j}{2i}
\end{aligned}$$

which is now in a form that can be partially differentiated, that is, $u(x, y)$ and $v(x, y)$ of the generalized circle expressed in terms of the spatial components x and y of the base circle **expk**

$$u(x, y) = \sum_{j=0}^{\infty} \left(\frac{1}{j!} f^{(j)}(0) \right) \frac{(x + i y)^j + (x - i y)^j}{2} \quad \text{and} \quad v(x, y) = \sum_{j=0}^{\infty} \left(\frac{1}{j!} f^{(j)}(0) \right) \frac{(x + i y)^j - (x - i y)^j}{2i}$$

For the Cauchy-Riemann and Laplace equations, the first partial derivatives are

$$\begin{aligned}
\text{Differentiating } u \text{ partially wrt } x \text{ for } u_x & \quad \frac{\partial}{\partial x} u(x, y) = \sum_{j=0}^{\infty} \left(\frac{1}{(j-1)!} f^{(j)}(0) \right) \frac{(x + i y)^{j-1} + (x - i y)^{j-1}}{2} \\
\text{Differentiating } v \text{ partially wrt } y \text{ for } v_y & \quad \frac{\partial}{\partial y} v(x, y) = \sum_{j=0}^{\infty} \left(\frac{1}{(j-1)!} f^{(j)}(0) \right) \frac{(x + i y)^{j-1} - (x - i y)^{j-1}}{2} \\
\text{First Cauchy-Riemann equation satisfied } u_x = v_y & \quad \frac{\partial}{\partial x} u(x, y) = \frac{\partial}{\partial y} v(x, y)
\end{aligned}$$

The x -component of the generalized circle (u) changes with respect to the x -component of the base **expk** circle (the $\hat{k}$ -hat exponential) in the same way that the conjugate y -component of the generalized circle (v) changes with respect to the conjugate y -component of the base **expk** circle (the $\hat{k}$ -hat exponential) since these functions are phase shifts of one another.

$$\begin{aligned}
\text{Differentiating } u \text{ partially wrt } y \text{ for } u_y & \quad \frac{\partial}{\partial y} u(x, y) = i \sum_{j=0}^{\infty} \left(\frac{1}{(j-1)!} f^{(j)}(0) \right) \frac{(x + i y)^{j-1} - (x - i y)^{j-1}}{2} \\
\text{Differentiating } v \text{ partially wrt } x \text{ for } -v_x & \quad \frac{\partial}{\partial x} v(x, y) = -i \sum_{j=0}^{\infty} \left(\frac{1}{(j-1)!} f^{(j)}(0) \right) \frac{(x + i y)^{j-1} - (x - i y)^{j-1}}{2} \\
\text{Second Cauchy-Riemann equation satisfied } u_y = -v_x & \quad \frac{\partial}{\partial y} u(x, y) = -\frac{\partial}{\partial x} v(x, y)
\end{aligned}$$

The x -component of the generalized circle (u) changes with respect to the conjugate y -component of the underlying circle (the $\hat{k}$ -hat exponential) in the same way that the conjugate y -component of the generalized circle (v) changes with respect to the x -component of the underlying circle (the $\hat{k}$ -hat exponential), again, because one is a phase shifts of the other.

These two equations mean that the generalized circle is differentiable (analytic) on the complex plane (the real R^2 x - y vector plane once a direction is specified). In other words, the derivative of the generalized circle is the same regardless of the direction in which

the limiting rate-of-change is taken [and hence differentiable on the complex (real x-y) plane]. This seems reasonable to me despite not yet doing this by varying theta. I think that the thetas cancel in the partials in which the Cauchy-Riemann equations are derived so if

$$f(z(x(\theta), y(\theta))) = u(x(\theta), y(\theta)) + i v(x(\theta), y(\theta))$$

then

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} & \frac{\partial f}{\partial(iy)} &= \frac{\partial u}{\partial(iy)} + i \frac{\partial v}{\partial(iy)} \\ \frac{\partial f}{\partial \theta} \frac{\partial \theta}{\partial x} &= \frac{\partial u}{\partial \theta} \frac{\partial \theta}{\partial x} + i \frac{\partial v}{\partial \theta} \frac{\partial \theta}{\partial x} & \text{and} & \frac{\partial f}{\partial \theta} \frac{\partial \theta}{\partial(iy)} &= \frac{\partial u}{\partial \theta} \frac{\partial \theta}{\partial(iy)} + i \frac{\partial v}{\partial \theta} \frac{\partial \theta}{\partial(iy)} \end{aligned}$$

and then equating real and imaginary parts for $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial(iy)}$ and hence the Cauchy-Riemann equations ($u_y = -v_x$ and $u_x = v_y$) showing it

is valid to ignore theta for proving that a generalized circle is analytic [differentiable] and harmonic.

Differentiating u again partially with respect to x for u_{xx}

$$\frac{\partial^2}{\partial x^2} u(x, y) = \sum_{j=0}^{\infty} \left(\frac{1}{(j-2)!} f^{(j)}(0) \right) \frac{(x+iy)^{j-2} + (x-iy)^{j-2}}{2}$$

Differentiating u again partially with respect to y for u_{yy}

$$\frac{\partial^2}{\partial y^2} u(x, y) = -\sum_{j=0}^{\infty} \left(\frac{1}{(j-2)!} f^{(j)}(0) \right) \frac{(x+iy)^{j-2} + (x-iy)^{j-2}}{2}$$

Laplace equation satisfied $u_{xx} + u_{yy} = 0$, u is harmonic

$$\frac{\partial^2}{\partial x^2} u(x, y) = -\frac{\partial^2}{\partial y^2} u(x, y) \quad \text{that is} \quad \frac{\partial^2}{\partial x^2} u(x, y) + \frac{\partial^2}{\partial y^2} u(x, y) = 0$$

Differentiating v again partially with respect to x for v_{xx}

$$\frac{\partial^2}{\partial x^2} v(x, y) = -i \sum_{j=0}^{\infty} \left(\frac{1}{(j-2)!} f^{(j)}(0) \right) \frac{(x+iy)^{j-2} - (x-iy)^{j-2}}{2}$$

Differentiating v again partially with respect to y for v_{yy}

$$\frac{\partial^2}{\partial y^2} v(x, y) = i \sum_{j=0}^{\infty} \left(\frac{1}{(j-2)!} f^{(j)}(0) \right) \frac{(x+iy)^{j-2} - (x-iy)^{j-2}}{2}$$

Laplace equation satisfied $v_{xx} + v_{yy} = 0$, v is harmonic

$$\frac{\partial^2}{\partial x^2} v(x, y) = -\frac{\partial^2}{\partial y^2} v(x, y) \quad \text{that is} \quad \frac{\partial^2}{\partial x^2} v(x, y) + \frac{\partial^2}{\partial y^2} v(x, y) = 0$$

Hence since $f(z) = u + iv$, the generalized circle $f(z)$ is hence harmonic on the real x-y plane with spin axis in the $\hat{k}$ direction (the latent z-axis) since it satisfies Laplace's equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0 \quad \text{or more fully} \quad \frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2} (\hat{k} \times)^2 = 0$$

noting that this, Laplace's equation, can be alternatively expressed after diadic expansion with $i = \hat{k} \times$ as

$$\det \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \hat{k} \times \right) f = 0 \quad \text{i.e.,} \quad \det \begin{pmatrix} \partial/\partial x & -\partial/\partial y & 0 \\ \partial/\partial y & \partial/\partial x & 0 \\ 0 & 0 & 1 \end{pmatrix} f = 0$$

Gauss row reduction (pivotal condensation) to Row Echelon Form and Reduced Row Echelon Form

This section provides a technique, which is a rigorous mathematical basis to the technique of Gauss row reduction to **row echelon form (ReF)** and **reduced row echelon form (RReF)**. This is essentially a method of row-reducing a matrix with complex numbers [basis cross-products]. I prefer not to use the term Gauss elimination since nothing is actually eliminated since any variable can always be reintroduced and are ultimately what are determined as a result of the pivotal condensations.

Gauss row reduction (also known as pivotal condensation) to row echelon form (ReF) and reduced row echelon form (RReF) are the most widely used techniques in linear algebra, including those in a vector volume R^3 . These techniques can be expressed as a chained equation of a series of complex numbers (if there were such a thing), that is, in terms of the basis curls/rotations [basis cross products] k -cross, j -cross, and i -cross. Because of the null rotational eigenvalue of the basis rotations (k -cross, j -cross, and i -cross), **left multiplications extract rows** and **right multiplications extract columns**. For example, $k \times$ extracts the first and second rows (the i -hat and j -hat rows) and permutes them. The second order basis curls extract single rows (or columns). For example, $k \times j \times$ extracts the 3rd row (the k -hat row) and places it in the 2nd row (the j -hat row). Similarly, $j \times k \times$ extracts the 2nd row (the j -hat row) and places it in the 3rd row (the k -hat row). This can then zero the leading term in the bottom row for example when scaled by the pivoting variables and added. This can then be continued as a Gauss reduction (pivotal condensation) to ReF or to RReF as a product series of k -cross, j -cross, and i -cross. The equivalent operators that have their rotational eigenvalues, the k -hat, j -hat, and i -hat exponentials at $\theta = \pi/2$ ($e^{(\pi/2)\hat{k} \times}$, $e^{(\pi/2)\hat{j} \times}$, and $e^{(\pi/2)\hat{i} \times}$) **swap rows** whilst leaving the row corresponding to their eigenvalue unchanged (not zeroed).

As an example,

$$\hat{j} \times \hat{k} \times \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ d & e & f \end{pmatrix}$$

so that

$$\left(1 - \frac{g}{d} \hat{j} \times \hat{k} \times\right) \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} a & b & c \\ d & e & f \\ 0 & h - ge/d & i - gf/d \end{pmatrix}$$

which demonstrates the first step, and product factor in the basis curls, in the reduction. The technique is laborious algebraically and long chained but I will be using this sophistication to build calculus diffusion equations. To demonstrated a full example, here for a simpler example, I will row reduce a 2by2 block in a 3by3 [since everything is in R^3] with a k -hat eigenvalue since the steps are familiar and most know the inverse and determinant so it is instantly recognisable. Noting again that if a matrix is 2by2 then the third dimension (axis) is latent for a vector volume. The matrix to be reduced, then inverted, is

$$\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Zeroing the ij' leading pivot in the leading column of the j -hat row with the operator $1 + \frac{c}{a} \hat{i} \times \hat{i} \times \hat{k} \times$

$$\left(1 + \frac{c}{a} \hat{i} \times \hat{i} \times \hat{k} \times\right) \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b & 0 \\ 0 & d - bc/a & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which is now in row echelon form. Zeroing the ji' term above the j -hat rows leading pivot with the operator $1 - \frac{ab}{ad - bc} \hat{i} \times \hat{i} \times \hat{k} \times$

$$\left(1 - \frac{ab}{ad - bc} \hat{i} \times \hat{i} \times \hat{k} \times\right) \left(1 + \frac{c}{a} \hat{i} \times \hat{i} \times \hat{k} \times\right) \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix} = \left(1 - \frac{ab}{ad - bc} \hat{i} \times \hat{i} \times \hat{k} \times\right) \begin{pmatrix} a & b & 0 \\ 0 & (ad - bc)/a & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a & 0 & 0 \\ 0 & (ad - bc)/a & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which is now in reduced row echelon form. The quaternion-like operator for a particular row can be used to return that row or its negative (or column if primed, i.e., transposed). Descaling the j -hat rows pivot to a one with the operator $1 + \left(\frac{a}{ad - bc} - 1\right) \hat{k} \times \hat{j} \times \hat{i} \times$

$$\left(1 + \left(\frac{a}{ad - bc} - 1\right) \hat{k} \times \hat{j} \times \hat{i} \times\right) \begin{pmatrix} a & 0 & 0 \\ 0 & (ad - bc)/a & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

noting that the complex number (third order real basis curl) that achieved this was the positive-quaternion-like operator $\hat{k} \times \hat{j} \times \hat{i} \times$ for j -hat (the j -cross is in the middle and the sign is determined by which of the others leads and which lags). Finally, zeroing the ii' leading pivot in the i -hat row with the operator $1 - \frac{a-1}{a} \hat{j} \times \hat{i} \times \hat{k} \times$

$$\left(1 - \frac{a-1}{a} \hat{j} \times \hat{i} \times \hat{k} \times\right) \begin{pmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which is the identity and fully reduced, noting that the operator for this factor was the negative-quaternion-like operator (or just as suitably the positive one) $\hat{j} \times \hat{i} \times \hat{k} \times$ which operated on the i -hat row (since i -cross was in the middle). In full, then with the four product factors in the basis curls, this reduction is

$$\left(1 - \frac{a-1}{a} \hat{j} \times \hat{i} \times \hat{k} \times\right) \left(1 + \left(\frac{a}{ad - bc} - 1\right) \hat{k} \times \hat{j} \times \hat{i} \times\right) \left(1 - \frac{ab}{ad - bc} \hat{i} \times \hat{i} \times \hat{k} \times\right) \left(1 + \frac{c}{a} \hat{i} \times \hat{i} \times \hat{k} \times\right) \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Note here the fact that the curls of the basis-hats (as complex-like numbers) do not have the rotational eigenvalues which is being used to do the row reduction. Also if the matrix is equated to the identity as an augmented matrix and this reduction is performed, then the identity of the augmented matrix is converted to the matrix's inverse. This then implies that applying these operators again then converts the identity further to the matrix inverse without having to augment an identity. In other words, the square of the row reducing polynomial of the basis-hats maps the matrix to its inverse. With the example,

$$\left(\left(1 - \frac{a-1}{a} \hat{j} \times \hat{i} \times \hat{k} \times \right) \left(1 + \left(\frac{a}{ad-bc} - 1 \right) \hat{k} \times \hat{j} \times \hat{i} \times \right) \left(1 - \frac{ab}{ad-bc} \hat{i} \times \hat{i} \times \hat{k} \times \right) \left(1 + \frac{c}{a} \hat{i} \times \hat{i} \times \hat{k} \times \right) \right)^2 \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} d/(ad-bc) & -c/(ad-bc) & 0 \\ -b/(ad-bc) & a/(ad-bc) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This is bona-fide mathematics and gives mathematical rigour to the technique of Gauss reduction (pivotal condensation) and obtaining the matrix inverse by the augmentation of an identity matrix before performing the reduction. In numerical applications, all of the coefficients of the row reducing basis curls [basis cross products] would be floating point arithmetic.

Hence, long polynomial chains in k -cross, j -cross, and i -cross are what reduces a matrix to the identity and further, is the mathematics that maps a matrix to its inverse. These polynomials in $\hat{k} \times$, $\hat{j} \times$, and $\hat{i} \times$ are roots of eigenvalue problems and so provide the **polynomial basis of dynamics** for diffusion (and inverse diffusion) calculus dynamics. These are also polynomials in $\hat{k} \times$, $\hat{j} \times$, and $\hat{i} \times$ when the coefficients are considered as powers and the roots are complex [real basis rotation operators]. I am hoping, if I get around to it, to turn these into higher order (long chain) diffusion equations that I am suspecting resolve into some very sophisticated dynamics, with no detail to the mathematics left unturned.

Mechanics of Row Reduction with Basis Curls (Basis Cross Products)

Noting that unprimed basis-hats are column vectors (the column basis) so when diadically expanded into a matrix basis with the primed basis-hats (the row basis) generate the rows, that is, the i -hat column vector generates the i -hat row when coupled with i' -hat, j' -hat, and k' -hat (unprimed coupled from the left with primed from the right for outer-product). Similarly, the j -hat column vector generates the j -hat row when coupled with i' -hat, j' -hat, and k' -hat and the k -hat column vector generates the k -hat row when coupled with i' -hat, j' -hat, and k' -hat.

Not relevant here because I have focused on row reduction, but if the reductions were column reductions (with right to left applications of k -cross, j -cross, and i -cross) then the primed basis-hats, which are row vectors, generate the columns when coupled (from the left for outer-product) with the unprimed basis-hats for the diadic matrix expansion to the matrix basis. So then the column reductions (pivotal condensations) would be in terms of zeroing and permuting the i' -hat, j' -hat, and k' -hat columns.

k -cross row reduction $\hat{k} \times A$: Left to right multiplication of k -cross to an R^3 matrix zeroes the third (k -hat) row, changes the sign of the second (j -hat) row and swaps it with the first (i -hat) row

$$\hat{k} \times \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} -d & -e & -f \\ a & b & c \\ 0 & 0 & 0 \end{pmatrix}$$

proof: If the matrix is $A = a\hat{i}\hat{i}' + b\hat{i}\hat{j}' + c\hat{i}\hat{k}' + d\hat{j}\hat{i}' + e\hat{j}\hat{j}' + f\hat{j}\hat{k}' + g\hat{k}\hat{i}' + h\hat{k}\hat{j}' + i\hat{k}\hat{k}'$ then the first, second, and third rows are $a\hat{i}\hat{i}' + b\hat{i}\hat{j}' + c\hat{i}\hat{k}'$, $d\hat{j}\hat{i}' + e\hat{j}\hat{j}' + f\hat{j}\hat{k}'$, and $g\hat{k}\hat{i}' + h\hat{k}\hat{j}' + i\hat{k}\hat{k}'$. Applying k -cross to the matrix

$$\begin{aligned}\hat{k} \times A &= \hat{k} \times (a\hat{i}\hat{i}' + b\hat{i}\hat{j}' + c\hat{i}\hat{k}' + d\hat{j}\hat{i}' + e\hat{j}\hat{j}' + f\hat{j}\hat{k}' + g\hat{k}\hat{i}' + h\hat{k}\hat{j}' + i\hat{k}\hat{k}') \\ &= \hat{k} \times (a\hat{i}\hat{i}' + b\hat{i}\hat{j}' + c\hat{i}\hat{k}') + \hat{k} \times (d\hat{j}\hat{i}' + e\hat{j}\hat{j}' + f\hat{j}\hat{k}') + \hat{k} \times (g\hat{k}\hat{i}' + h\hat{k}\hat{j}' + i\hat{k}\hat{k}')\end{aligned}$$

where the rows have been separated into individual terms of the three matrix basis components of each row. For the first row $a\hat{i}\hat{i}' + b\hat{i}\hat{j}' + c\hat{i}\hat{k}'$ applying k -cross

$$\begin{aligned}\hat{k} \times (a\hat{i}\hat{i}' + b\hat{i}\hat{j}' + c\hat{i}\hat{k}') &= a(\hat{k} \times \hat{i})\hat{i}' + b(\hat{k} \times \hat{i})\hat{j}' + c(\hat{k} \times \hat{i})\hat{k}' \\ &= a\hat{j}\hat{i}' + b\hat{j}\hat{j}' + c\hat{j}\hat{k}'\end{aligned}$$

which is now, by the properties of k -cross, the second row. That is, the first row of A is now the second row of $\hat{k} \times A$. For the second row $d\hat{j}\hat{i}' + e\hat{j}\hat{j}' + f\hat{j}\hat{k}'$, applying k -cross

$$\begin{aligned}\hat{k} \times (d\hat{j}\hat{i}' + e\hat{j}\hat{j}' + f\hat{j}\hat{k}') &= d(\hat{k} \times \hat{j})\hat{i}' + e(\hat{k} \times \hat{j})\hat{j}' + f(\hat{k} \times \hat{j})\hat{k}' \\ &= -d\hat{i}\hat{i}' - e\hat{i}\hat{j}' - f\hat{i}\hat{k}'\end{aligned}$$

which now comprises the first row of $\hat{k} \times A$ and is sign changed throughout. That is, the second row of A has its sign changed and then permutes row position to the first row of $\hat{k} \times A$. For the third row $g\hat{k}\hat{i}' + h\hat{k}\hat{j}' + i\hat{k}\hat{k}'$, applying k -cross

$$\begin{aligned}\hat{k} \times (g\hat{k}\hat{i}' + h\hat{k}\hat{j}' + i\hat{k}\hat{k}') &= g(\hat{k} \times \hat{k})\hat{i}' + h(\hat{k} \times \hat{k})\hat{j}' + i(\hat{k} \times \hat{k})\hat{k}' \\ &= 0\hat{k}\hat{i}' + 0\hat{k}\hat{j}' + 0\hat{k}\hat{k}'\end{aligned}$$

which is a third row of zeros. That is, the third row of A is zeroed which comprises a null row of zeros for $\hat{k} \times A$ because of the null rotational eigenvector of k -cross. **End of proof.**

It is the null rotational eigenvector of k -cross that is used to row reduce. This can then be scaled and added to A to zero a leading pivot. Note that similar properties hold for reducing to column echelon forms by right to left multiplications by the basis-rotations (which then zero columns rather than rows).

j -cross- k -cross row reduction $\hat{j} \times \hat{k} \times A$: Left to right multiplication of j -cross- k -cross to an R^3 matrix takes the second row and places it in the third row and zeroes the other two rows

$$\hat{j} \times \hat{k} \times \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ d & e & f \end{pmatrix}$$

proof: If the matrix is $A = a\hat{i}\hat{i}' + b\hat{i}\hat{j}' + c\hat{i}\hat{k}' + d\hat{j}\hat{i}' + e\hat{j}\hat{j}' + f\hat{j}\hat{k}' + g\hat{k}\hat{i}' + h\hat{k}\hat{j}' + i\hat{k}\hat{k}'$ then the first, second and third rows are $a\hat{i}\hat{i}' + b\hat{i}\hat{j}' + c\hat{i}\hat{k}'$, $d\hat{j}\hat{i}' + e\hat{j}\hat{j}' + f\hat{j}\hat{k}'$, and $g\hat{k}\hat{i}' + h\hat{k}\hat{j}' + i\hat{k}\hat{k}'$. Applying j -cross- k -cross to the matrix is then

$$\begin{aligned}\hat{j} \times \hat{k} \times A &= \hat{j} \times \hat{k} \times (a\hat{i}\hat{i}' + b\hat{i}\hat{j}' + c\hat{i}\hat{k}' + d\hat{j}\hat{i}' + e\hat{j}\hat{j}' + f\hat{j}\hat{k}' + g\hat{k}\hat{i}' + h\hat{k}\hat{j}' + i\hat{k}\hat{k}') \\ &= \hat{j} \times \hat{k} \times (a\hat{i}\hat{i}' + b\hat{i}\hat{j}' + c\hat{i}\hat{k}') + \hat{j} \times \hat{k} \times (d\hat{j}\hat{i}' + e\hat{j}\hat{j}' + f\hat{j}\hat{k}') + \hat{j} \times \hat{k} \times (g\hat{k}\hat{i}' + h\hat{k}\hat{j}' + i\hat{k}\hat{k}')\end{aligned}$$

where the rows have been separated into individual terms of the three matrix basis components of each row similarly as for a single k -cross. For the first row $a\hat{i}\hat{i}' + b\hat{i}\hat{j}' + c\hat{i}\hat{k}'$ applying j -cross- k -cross

$$\begin{aligned}\hat{j} \times \hat{k} \times (a\hat{i}\hat{i}' + b\hat{i}\hat{j}' + c\hat{i}\hat{k}') &= a(\hat{j} \times \hat{k} \times \hat{i})\hat{i}' + b(\hat{j} \times \hat{k} \times \hat{i})\hat{j}' + c(\hat{j} \times \hat{k} \times \hat{i})\hat{k}' \\ &= a(\hat{j} \times \hat{j})\hat{i}' + b(\hat{j} \times \hat{j})\hat{j}' + c(\hat{j} \times \hat{j})\hat{k}' \\ &= 0\hat{j}\hat{i}' + 0\hat{j}\hat{j}' + 0\hat{j}\hat{k}'\end{aligned}$$

which is now, by the properties of k -cross, the second row which then is subsequently a row of zeros by the properties of j -cross from right to left evaluation of the second order mixed-basis-cross. That is, the first row of A is now the second row of $\hat{j} \times \hat{k} \times A$ which is then zeroed. For the second row $d\hat{j}\hat{i}' + e\hat{j}\hat{j}' + f\hat{j}\hat{k}'$, applying j -cross- k -cross

$$\begin{aligned}\hat{j} \times \hat{k} \times (d\hat{j}\hat{i}' + e\hat{j}\hat{j}' + f\hat{j}\hat{k}') &= d(\hat{j} \times \hat{k} \times \hat{j})\hat{i}' + e(\hat{j} \times \hat{k} \times \hat{j})\hat{j}' + f(\hat{j} \times \hat{k} \times \hat{j})\hat{k}' \\ &= -d(\hat{j} \times \hat{i})\hat{i}' - e(\hat{j} \times \hat{i})\hat{j}' - f(\hat{j} \times \hat{i})\hat{k}' \\ &= d\hat{k}\hat{i}' + e\hat{k}\hat{j}' + f\hat{k}\hat{k}'\end{aligned}$$

which now comprises the third row of $\hat{j} \times \hat{k} \times A$. That is, the second row of A is now the third row of $\hat{j} \times \hat{k} \times A$. For the third row $g\hat{k}\hat{i}' + h\hat{k}\hat{j}' + i\hat{k}\hat{k}'$, applying j -cross- k -cross

$$\begin{aligned}\hat{j} \times \hat{k} \times (g\hat{k}\hat{i}' + h\hat{k}\hat{j}' + i\hat{k}\hat{k}') &= g(\hat{j} \times \hat{k} \times \hat{k})\hat{i}' + h(\hat{j} \times \hat{k} \times \hat{k})\hat{j}' + i(\hat{j} \times \hat{k} \times \hat{k})\hat{k}' \\ &= 0(\hat{j} \times \hat{k})\hat{i}' + 0(\hat{j} \times \hat{k})\hat{j}' + 0(\hat{j} \times \hat{k})\hat{k}' \\ &= 0\hat{i}\hat{i}' + 0\hat{i}\hat{j}' + 0\hat{i}\hat{k}'\end{aligned}$$

which zeroes the third row by k -cross and the places the row of zeros in the first row by j -cross. That is, the third row of A is zeroed which comprises a null row of zeros and is then permuted to the first row, that is, becomes a first row of zeros for $\hat{j} \times \hat{k} \times A$. **End of proof.**

Both of these proofs are easily shown by matrix multiplication on a TI or with MATLAB for instance. These are concepts that work. The difficulty is choosing the basis curls [basis cross products] that achieve the desired row reduction [which is not trivial although can be achieved by eigenvalues and then the minimum basis factorization deduced]. Doing this well would probably come with experience. I have placed all of the basis-cross operators in diadically expanded form on the next few pages and these can be matrix multiplied for all of the results on a programmable calculator which is how I have been working the theory out [in this section and quite a few of the other sections].

Diadically expanded first order and second order basis curls [basis-cross-products]

Note further, and kind of interestingly, that vector and matrix bases are essentially systems of ones and zeros. This means that we are getting all of this dynamics, ultimately, through **configurations of digital systems**. I think that is very interesting, that is, that a basis is a form of digital system. In the operators (combinations of the standard basis vectors appended with $\times$), there is also negative one. Basis components are single component unit vectors with a single positive one. It is the basis-curls [basis cross products] k -cross, j -cross, and i -cross that introduce the negative ones which is also very interesting. **Bases of $\mathbf{R}^3$ really are digital systems**. Basis-crosses extend the two component digital systems of the basis to the three component (trigital) system of one, zero, and negative one which is convolving one digital system of **zero and one** into another digital system of **positive and negative**. Interesting. Anyway...

The diadically expanded operators: the first three are literally complex numbers [real basis curls]. Those with single positive ones, unit matrices, have the exact same form [when positive] as matrix basis components but these are operators, that is, they are defined specifically to multiply (operate on) vectors such as position or velocity vectors for example (since the column basis of a vector has been assumed in order to diadically expand the operators). The diadically expanded operators are

First order

$$\hat{i} \times = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad \hat{j} \times = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad \hat{k} \times = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Second order

$$\begin{aligned} \hat{i} \times \hat{i} \times &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \text{ double cross } \hat{j} \times \hat{i} \times = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \hat{k} \times \hat{i} \times &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \hat{i} \times \hat{j} \times &= \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \hat{j} \times \hat{j} \times &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \text{ double cross } & \hat{k} \times \hat{j} \times &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} & \text{nb } (\hat{k} \times \hat{j} \times)^T &= \hat{j} \times \hat{k} \times \\ \hat{i} \times \hat{k} \times &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} & \hat{j} \times \hat{k} \times &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} & \hat{k} \times \hat{k} \times &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ double cross} \end{aligned}$$

Third order

$$\hat{i} \times \hat{j} \times \hat{k} \times = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \hat{j} \times \hat{k} \times \hat{i} \times = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \hat{k} \times \hat{i} \times \hat{j} \times = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\hat{k} \times \hat{j} \times \hat{i} \times = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\hat{i} \times \hat{k} \times \hat{j} \times = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\hat{j} \times \hat{i} \times \hat{k} \times = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

as I said earlier in this document, "it seems simple and it is simple", but it works! Easy to program as well. I'm doing it all on my Texas Instruments workhorse... I used to do mathematical and statistical computing on my laptop but now it is way too unreliable for anything other than word processing. Note that the **second order basis curls** [basis cross products] are the matrix product of two of the **first order basis curls** [basis cross products], and similarly the **third order basis curls** [basis cross products], for example, $\hat{k} \times \hat{j} \times \hat{i} \times$ is the matrix product of $\hat{k} \times$, $\hat{j} \times$, and $\hat{i} \times$. Note further the set most important from complex analysis is i , $i^2 = -1$, $i^3 = -i$, and $i^4 = +1$

$$\hat{k} \times = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$(\hat{k} \times)^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$(\hat{k} \times)^3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$(\hat{k} \times)^4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} (\neq I)$$

noting that $(\hat{k} \times)^4$ is not the identity ($(\hat{k} \times)^4 \neq +1I$) because of the null k -hat eigenvalue (which is otherwise populated by the k -hat component of a real part). Note also that in general, not ignoring latent k -hat, $(\hat{k} \times)^2 \neq -1$ for the same reason, that is, the square of what was known as a complex number is not equal to negative one because it is only so when operating on i -hat and j -hat, and not when operating on k -hat. If a long chain combination of basis curls has the same diadically expanded matrix as a simpler combination, then the complicated combination can be replaced by the simpler combination [which has relevance to efficient Rubiks cube solving]. There is a minimal polynomial theorem for the **factorization of a matrix** as a **polynomial function of basis curls** as I will discuss shortly.

In my opinion, for this method to be fully functional, it needs a further expansion to a cubic tensor so that it works one-for-one with a Rubik's cube. This will be when this theory reaches its full potential. This will possibly be a 3 component linear combination of 3 matrices weighting a double-primed-into-the-page vector basis for a triadic vector expansion to the 27 component cubic tensor basis.

Matrix Basis Factorization with Basis Rotations (basis-cross-products)

If A is a matrix and $A^{-1}A$ equals the identity then A has an inverse and this inverse is A^{-1} .

So if an operator converts A to the identity, then it must be its inverse. Therefore, a row-reducing polynomial in k -cross, j -cross, and/or i -cross that converts a matrix to the identity must be its inverse. That is, if $p(\hat{i} \times, \hat{j} \times, \hat{k} \times)$ is a polynomial and when applied to matrix A results in $p(\hat{i} \times, \hat{j} \times, \hat{k} \times)A = I$, then

$$p(\hat{i} \times, \hat{j} \times, \hat{k} \times) = A^{-1} \text{ and equivalently } A = p^{-1}(\hat{i} \times, \hat{j} \times, \hat{k} \times).$$

So if p applied to A returns I , this (A^{-1}) is what is obtained after diadic expansion of $p(\hat{i} \times, \hat{j} \times, \hat{k} \times)$ into a matrix basis.

So if $p(\hat{i} \times, \hat{j} \times, \hat{k} \times)$ is known for A then the inverse is obtained by either **(1)** squaring p and multiplying (as an operator) by A (the operand) or **(2)** diadically expanding p [with an assumed column vector basis of an arbitrary operand]. Further, if p applied to A returns I , then p^{-1} is a polynomial factorisation of the matrix A . This polynomial factorisation of the matrix A , then, when diadically expanded returns A in the matrix form.

Somewhat of a tautology, somewhat not. What is interesting is that a matrix, not surprisingly, can be written out in full as a scalar weighted linear combination of the matrix basis components. The interesting thing, then, is that this polynomial is factored into roots in the basis curls/rotations, k -cross, j -cross, and i -cross. From such an equation, [complex basis rotation] roots can be extracted. These roots can then be used to define a matrix. They can also be roots for rotational dynamics or, for example, divergence double-crosses.

Hence any matrix can be written as basis variables (the matrix basis weighted by the component scalars) and then factored into roots in basis rotations [basis cross products] where this factorisation is determined [as an inverse and then inverted back again] as the row reducing polynomial as a factored product of complex [real rotational] roots that returns the identity.

What is interesting to me is that the roots in the basis rotations don't require a calculus D although this is what they essentially are. Therefore, these can be placed into a differential equation without having to up the order of the differential equation. Although this is the ultimate goal, it also raises the mathematical difficulties in resolving the differential equation [then of much higher order]. I am thinking now that I can put long chains of these into first order equations (Newton's Law of Cooling) work through all the maths at first order and then obtain the results of rotational and divergence dynamics of large order differential systems [again, which is the goal because this is where the sophistication is].

Say the matrix that is to be row reduced and then inverted is

$$\begin{pmatrix} 1 & -a & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This is equivalent to the polynomial in k -cross, $1 + a\hat{k} \times$, when this polynomial is diadically expanded. Therefore **(1)** $(1 + a\hat{k} \times)^{-1}$ will row reduce the matrix to the identity (when applied to A) and **(2)** the square $(1 + a\hat{k} \times)^{-2}$ will return the inverse (when applied to A), since $(1 + a\hat{k} \times)^{-1}$ is the matrices' inverse

$$(1 + a\hat{k} \times)^{-1} A = (1 + a\hat{k} \times)^{-1} \begin{pmatrix} 1 & -a & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$(1 + a\hat{k} \times)^{-2} A = (1 + a\hat{k} \times)^{-2} \begin{pmatrix} 1 & -a & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1/(1+a^2) & a/(1+a^2) & 0 \\ -a/(1+a^2) & 1/(1+a^2) & 0 \\ 0 & 0 & 1 \end{pmatrix} = (1 + a\hat{k} \times)^{-1} = A^{-1}$$

Rationalising the complex denominator with its conjugate (and subsequently diadically expanding)

$$(1 + a \hat{k} \times)^{-1} = \frac{1}{1 + a \hat{k} \times} = \frac{1}{1 + a \hat{k} \times} \frac{1 - a \hat{k} \times}{1 - a \hat{k} \times} = \frac{1 - a \hat{k} \times}{1 + a^2 \hat{k} \times \hat{k} \times} = \frac{1 - a \hat{k} \times}{1 + a^2}$$

as a rationalized complex quantity which works the same as standard complex number theory but is only applicable to the i -hat and j -hat directions (it is unity when numerator and denominator are multiplied by k -hat). Diadically expanding

$$\frac{1 - a \hat{k} \times}{1 + a^2} = \begin{pmatrix} 1/(1+a^2) & a/(1+a^2) & 0 \\ -a/(1+a^2) & 1/(1+a^2) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which shows that it is indeed the matrix inverse [the rationalized conjugate]. Therefore, the inverse of the row reducing polynomial (which is the inverse) is the basis-cross polynomial factorisation of the matrix. Since there are many ways to row reduce a matrix to RREF and further to the identity, such a polynomial is not unique, but there is a minimal polynomial which does this so there will always exist a minimal polynomial in the rotation operators (the basis-crosses) that is unique. For this example, $(1 + a \hat{k} \times)^{-1}$ is the minimal row reducing polynomial.

Here is the mechanics of the row reducing polynomial in k -cross, remembering that k -cross zeroes the third (k^{th}) row, changes the sign of the second row and then permutes it with the first row (permutes the first and second rows changing the sign of the second row which is then the first row)

$$\hat{k} \times \begin{pmatrix} 1 & -a & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -a & -1 & 0 \\ 1 & -a & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

so that

$$\begin{aligned} (1 + a \hat{k} \times)^{-1} \begin{pmatrix} 1 & -a & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} &= \frac{1 - a \hat{k} \times}{1 + a^2} \begin{pmatrix} 1 & -a & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \frac{1}{1 + a^2} \begin{pmatrix} 1 & -a & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \frac{a}{1 + a^2} \hat{k} \times \begin{pmatrix} 1 & -a & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \frac{1}{1 + a^2} \begin{pmatrix} 1 & -a & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \frac{a}{1 + a^2} \begin{pmatrix} -a & -1 & 0 \\ 1 & -a & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{1}{1 + a^2} \begin{pmatrix} 1 & -a & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \frac{1}{1 + a^2} \begin{pmatrix} -a^2 & -a & 0 \\ a & -a^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \frac{1}{1 + a^2} \begin{pmatrix} 1 + a^2 & 0 & 0 \\ 0 & 1 + a^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{since} \quad \frac{1}{1 + a^2 (\hat{k} \times)^2} \hat{k} \hat{k}' = \hat{k} \hat{k}' \\ &= I \end{aligned}$$

which has hence row reduced the matrix to the identity by what is essentially pivotal condensation (Gauss reduction) by utilizing the properties of complex numbers, here, the complete row reduction performed by the properties of $i = \hat{k} \times$. Note that for this example, the row reducing polynomial $(1 + a \hat{k} \times)^{-1}$ is a low-pass transfer function; an RC circuit with the voltage drop taken over the capacitor [with an ac source $\Rightarrow$ hence an ac transfer function]. Note also that Euler's equation for the complex exponential [the k -hat exponential] is the minimal factorisation polynomial in k -cross for the rotation matrix. Any 3 by 3 matrix is probably a product of

rotation and divergence matrices which can then factorize into a minimum polynomial in k -cross, j -cross, and/or i -cross. This will be a factorisation in the divergences and rotational roots and be a full form of the Fundamental Theorem of Algebra. This gives full meaning to the rotational basis operators over and above the insufficient and the less than satisfying definition $i = \sqrt{-1}$. Despite all of the maths of complex theory holding superbly well for the Argand plane, it was very vexing given that it was clear that there was more to the story; the most vexing issues for me were (and are), in addition to what $i = \sqrt{-1}$ actually was, are the mathematical dead ends of divergence of curl (nabla-dot nabla-cross) and curl of gradient (nabla-cross nabla[grad]), and infinite limits on, and hence non-converging, Fourier integrals (the latter issue now solved with theory regarding $i = \hat{k} \times$).

For the details of the diadic expansion of the minimum polynomial $(1 + a\hat{k} \times)^{-1}$ for inversion of the stipulated matrix operator of k -cross, in the i -hat direction

$$\frac{1 - a\hat{k} \times}{1 + a^2} \hat{i} = \frac{\hat{i} - a\hat{k} \times \hat{i}}{1 + a^2} = \hat{i} \frac{1}{1 + a^2} - \hat{j} \frac{a}{1 + a^2} + \hat{k} 0$$

j -hat direction

$$\frac{1 - a\hat{k} \times}{1 + a^2} \hat{j} = \frac{\hat{j} - a\hat{k} \times \hat{j}}{1 + a^2} = \hat{i} \frac{a}{1 + a^2} + \hat{j} \frac{1}{1 + a^2} + \hat{k} 0$$

For the k -hat direction, because the denominator is still latently dependent on $\hat{k} \times$, it is not true that

$$\frac{1 - a\hat{k} \times}{1 + a^2} \hat{k} = \frac{\hat{k} - a\hat{k} \times \hat{k}}{1 + a^2} = \frac{\hat{k}}{1 + a^2} = \hat{i} 0 + \hat{j} 0 + \hat{k} \frac{1}{1 + a^2}$$

With a power series expansion

$$(1 + a\hat{k} \times)^{-1} = 1 - a\hat{k} \times + a^2(\hat{k} \times)^2 - a^3(\hat{k} \times)^3 + a^4(\hat{k} \times)^4 - a^5(\hat{k} \times)^5 + \dots$$

applying the operator to k -hat

$$\begin{aligned} (1 + a\hat{k} \times)^{-1} \hat{k} &= \left(1 - a\hat{k} \times + a^2(\hat{k} \times)^2 - a^3(\hat{k} \times)^3 + a^4(\hat{k} \times)^4 - a^5(\hat{k} \times)^5 + \dots\right) \hat{k} \\ &= \hat{k} - a\hat{k} \times \hat{k} + a^2(\hat{k} \times)^2 \hat{k} - a^3(\hat{k} \times)^3 \hat{k} + a^4(\hat{k} \times)^4 \hat{k} - a^5(\hat{k} \times)^5 \hat{k} + \dots \\ &= \hat{i} 0 + \hat{j} 0 + \hat{k} 1 \end{aligned}$$

which is, rather, the unity k -hat eigenvector. For i -hat and j -hat, the results are as earlier, unlike for k -hat for which the complex rationalization gave misleading results when applied to the k -hat direction

$$\begin{aligned}
(1 + a \hat{k} \times)^{-1} \hat{i} &= \left(1 - a \hat{k} \times + a^2 (\hat{k} \times)^2 - a^3 (\hat{k} \times)^3 + a^4 (\hat{k} \times)^4 - a^5 (\hat{k} \times)^5 + \dots \right) \hat{i} \\
&= \hat{i} - a \hat{k} \times \hat{i} + a^2 (\hat{k} \times)^2 \hat{i} - a^3 (\hat{k} \times)^3 \hat{i} + a^4 (\hat{k} \times)^4 \hat{i} - a^5 (\hat{k} \times)^5 \hat{i} + \dots \\
&= \hat{i} - \hat{j} a - \hat{i} a^2 + \hat{j} a^3 + \hat{i} a^4 - \hat{j} a^5 + \dots \\
&= \hat{i} (1 - a^2 + a^4 - \dots) + \hat{j} (-a + a^3 - a^5 + \dots) + \hat{k} 0 \\
&= \hat{i} (1 + a^2)^{-1} - \hat{j} a (1 + a^2)^{-1} + \hat{k} 0
\end{aligned}$$

and

$$\begin{aligned}
(1 + a \hat{k} \times)^{-1} \hat{j} &= \left(1 - a \hat{k} \times + a^2 (\hat{k} \times)^2 - a^3 (\hat{k} \times)^3 + a^4 (\hat{k} \times)^4 - a^5 (\hat{k} \times)^5 + \dots \right) \hat{j} \\
&= \hat{j} - a \hat{k} \times \hat{j} + a^2 (\hat{k} \times)^2 \hat{j} - a^3 (\hat{k} \times)^3 \hat{j} + a^4 (\hat{k} \times)^4 \hat{j} - a^5 (\hat{k} \times)^5 \hat{j} + \dots \\
&= \hat{j} + \hat{i} a - \hat{j} a^2 - \hat{i} a^3 + \hat{j} a^4 + \hat{i} a^5 + \dots \\
&= \hat{i} (a - a^3 + a^5 - \dots) + \hat{j} (1 - a^2 + a^4 - \dots) + \hat{k} 0 \\
&= \hat{i} a (1 + a^2)^{-1} + \hat{j} (1 + a^2)^{-1} + \hat{k} 0
\end{aligned}$$

as earlier by rationalizing the denominator with standard complex theory. It works exactly the same until k -hat is considered for a full 3 by 3 matrix basis. The three column components from each basis direction are

$$(1 + a \hat{k} \times)^{-1} \hat{i} = \hat{i} (1 + a^2)^{-1} - \hat{j} a (1 + a^2)^{-1} + \hat{k} 0$$

$$(1 + a \hat{k} \times)^{-1} \hat{j} = \hat{i} a (1 + a^2)^{-1} + \hat{j} (1 + a^2)^{-1} + \hat{k} 0$$

$$(1 + a \hat{k} \times)^{-1} \hat{k} = \hat{i} 0 + \hat{j} 0 + \hat{k} 1$$

which hence diadically expands into

$$\begin{pmatrix} 1/(1+a^2) & a/(1+a^2) & 0 \\ -a/(1+a^2) & 1/(1+a^2) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which is the inverse of $1 + a \hat{k} \times$

$$(1 + a \hat{k} \times)^{-1} = \begin{pmatrix} 1 & -a & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1/(1+a^2) & a/(1+a^2) & 0 \\ -a/(1+a^2) & 1/(1+a^2) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

by standard matrix inversion of the operator and hence shows that they are equivalent.

Here is one (of several) method(s) of showing the obvious result that the **Euler form** of the **k -hat conjugate exponential** is the minimal row reducing polynomial for the **k -hat exponential** (as the minimum row reducing polynomial for the rotation matrix). The k -hat exponential is

$$e^{\theta \hat{k} \otimes} = \mathbf{cosk}\theta + \mathbf{sink}\theta \hat{k} \times = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and the k -hat conjugate is

$$e^{-\theta \hat{k} \otimes} = (\mathbf{cosk}\theta + \mathbf{sink}\theta \hat{k} \times)^{-1} = (\mathbf{cosk}\theta - \mathbf{sink}\theta \hat{k} \times)$$

Using the first order polynomial in k -cross for the k -hat conjugate to row reduce

$$\begin{aligned} (\mathbf{cosk}\theta - \mathbf{sink}\theta \hat{k} \times) \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} &= (\mathbf{cosk}\theta) \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} - (\mathbf{sink}\theta \hat{k} \times) \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \cos \theta \mathbf{cosk}\theta & -\sin \theta \mathbf{cosk}\theta & 0 \\ \sin \theta \mathbf{cosk}\theta & \cos \theta \mathbf{cosk}\theta & 0 \\ 0 & 0 & \mathbf{cosk}\theta \end{pmatrix} + \begin{pmatrix} \cos \theta \mathbf{sink}\theta \hat{k} \times & -\sin \theta \mathbf{sink}\theta \hat{k} \times & 0 \\ \sin \theta \mathbf{sink}\theta \hat{k} \times & \cos \theta \mathbf{sink}\theta \hat{k} \times & 0 \\ 0 & 0 & \mathbf{sink}\theta \hat{k} \times \end{pmatrix} \\ &= \begin{pmatrix} \cos \theta \mathbf{cosk}\theta & -\sin \theta \mathbf{cosk}\theta & 0 \\ \sin \theta \mathbf{cosk}\theta & \cos \theta \mathbf{cosk}\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} \cos \theta \mathbf{sink}\theta \hat{k} \times & -\sin \theta \mathbf{sink}\theta \hat{k} \times & 0 \\ \sin \theta \mathbf{sink}\theta \hat{k} \times & \cos \theta \mathbf{sink}\theta \hat{k} \times & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

since $(\mathbf{cosk}\theta \hat{k}) \hat{k}' = (\hat{k}1) \hat{k}' = \hat{k} \hat{k}' 1$ and $(\mathbf{sink}\theta \hat{k} \times \hat{k}) \hat{k}' = (0 \hat{k}) \hat{k}' = \hat{k} \hat{k}' 0$

$$= \begin{pmatrix} \cos^2 \theta & -\sin \theta \cos \theta & 0 \\ \sin \theta \cos \theta & \cos^2 \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} \cos \theta \mathbf{sink}\theta \hat{k} \times & -\sin \theta \mathbf{sink}\theta \hat{k} \times & 0 \\ \sin \theta \mathbf{sink}\theta \hat{k} \times & \cos \theta \mathbf{sink}\theta \hat{k} \times & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

since $(\mathbf{cosk}\theta) \hat{i} \hat{i}' = (\hat{i} \cos \theta) \hat{i}' = \hat{i} (\cos \theta \hat{i}') = \hat{i} \hat{i}' \cos \theta$ and similarly with **cosk** for the $\hat{i} \hat{j}'$, $\hat{j} \hat{i}'$, $\hat{j} \hat{j}'$ components of the leading matrix

$$= \begin{pmatrix} \cos^2 \theta & -\sin \theta \cos \theta & 0 \\ \sin \theta \cos \theta & \cos^2 \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} \sin^2 \theta & \sin \theta \cos \theta & 0 \\ -\sin \theta \cos \theta & \sin^2 \theta & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

since $(\mathbf{sink}\theta \hat{k} \times) \hat{i} \hat{i}' = \mathbf{sink}\theta (\hat{k} \times \hat{i}) \hat{i}' = (\mathbf{sink}\theta \hat{j}) \hat{i}' = (\hat{j} \sin \theta) \hat{i}' = \hat{j} (\sin \theta \hat{i}') = \hat{j} \hat{i}' \sin \theta$ which shifts the position of this component to the next column. The basis calculations are similar for the other **sink** components $(\mathbf{sink}\theta \hat{k} \times) \hat{i} \hat{j}'$, $(\mathbf{sink}\theta \hat{k} \times) \hat{j} \hat{i}'$, and $(\mathbf{sink}\theta \hat{k} \times) \hat{j} \hat{j}'$ for the components of the lagging matrix. So adding then gives the identity

$$= (\mathbf{cosk}\theta - \mathbf{sink}\theta \hat{k} \times) \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ = I$$

and therefore, not surprisingly, $\mathbf{cosk}\theta - \mathbf{sink}\theta \hat{k} \times$ (conjugate **expk**) is the row reducing polynomial to reduced-row-echelon-form for the k -hat exponential **expk**. Tautology or not, it works. This is a union of complex theory (real vectors of the standard basis appended with $\times$) with R^3 linear algebra (as compositions of the standard basis).

Some interesting further configurations, which are ultimately systems of dynamics, are obtained by conjugating the rotational roots. For example, with the two conjugates

$$1 + a \hat{k} \times = \begin{pmatrix} 1 & -a & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad 1 - a \hat{k} \times = \begin{pmatrix} 1 & a & 0 \\ -a & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

conjugate into the diagonal

$$(1 + a \hat{k} \times)(1 - a \hat{k} \times) = \begin{pmatrix} 1 + a^2 & 0 & 0 \\ 0 & 1 + a^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which is divergence-like. I know how to turn this into something very special but I defer this because I am focusing solely on the basis-rotations. Similarly,

$$(1 + b \hat{j} \times)(1 - b \hat{j} \times) = \begin{pmatrix} 1 + b^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 + b^2 \end{pmatrix} \quad \text{and} \quad (1 + c \hat{i} \times)(1 - b c \hat{i} \times) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 + c^2 & 0 \\ 0 & 0 & 1 + c^2 \end{pmatrix}$$

so that

$$(1 + a \hat{k} \times)(1 - a \hat{k} \times) (1 + b \hat{j} \times)(1 - b \hat{j} \times) (1 + c \hat{i} \times)(1 - b c \hat{i} \times) = \begin{pmatrix} (1 + a^2)(1 + b^2) & 0 & 0 \\ 0 & (1 + a^2)(1 + c^2) & 0 \\ 0 & 0 & (1 + b^2)(1 + c^2) \end{pmatrix}$$

which I think will correspond with moments of inertia although what I am really hoping for is dipole antenna radiation patterns. This corresponds to an order 6 differential equation that can be either all-poles or all-zeros. For my interests, it will be all-poles [for dynamics rather than vibration and smoothing].

Cayley-Hamilton theorem: why does a matrix satisfy its own characteristic equation?

The Cayley-Hamilton theorem is a very important theorem relating to linear systems of differential equations and the matrix exponential which very obviously has overwhelming relevance to k -hat exponentials and therefore very relevant to this document.

The Cayley-Hamilton theorem states that a matrix satisfies its own characteristic equation $\phi(A) = 0$. Here is a proof.

Proof $\phi(A) = 0$: The characteristic equation of a matrix A is defined as the determinant $\phi(\lambda) = \det(A - \lambda I)$.

If the matrix is $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ or in terms of the matrix basis $A = a\hat{i}\hat{i}' + b\hat{i}\hat{j}' + c\hat{i}\hat{k}' + d\hat{j}\hat{i}' + e\hat{j}\hat{j}' + f\hat{j}\hat{k}' + g\hat{k}\hat{i}' + h\hat{k}\hat{j}' + i\hat{k}\hat{k}'$

Then

$$\phi(\lambda) = \det(A - \lambda I) = \begin{vmatrix} a - \lambda & b & c \\ d & e - \lambda & f \\ g & h & i - \lambda \end{vmatrix}$$

that is,

$$\phi(\lambda) = \left| (a - \lambda)\hat{i}\hat{i}' + b\hat{i}\hat{j}' + c\hat{i}\hat{k}' + d\hat{j}\hat{i}' + (e - \lambda)\hat{j}\hat{j}' + f\hat{j}\hat{k}' + g\hat{k}\hat{i}' + h\hat{k}\hat{j}' + (i - \lambda)\hat{k}\hat{k}' \right|$$

so what is to be shown is that $\phi(A) = 0$

$$\phi(A) = \left| (a - A)\hat{i}\hat{i}' + b\hat{i}\hat{j}' + c\hat{i}\hat{k}' + d\hat{j}\hat{i}' + (e - A)\hat{j}\hat{j}' + f\hat{j}\hat{k}' + g\hat{k}\hat{i}' + h\hat{k}\hat{j}' + (i - A)\hat{k}\hat{k}' \right| = 0$$

This is the determinant of the matrix A nested within itself down the main diagonal. Considering the top left hand component, the $\hat{i}\hat{i}'$ component

$$(a - A)\hat{i}\hat{i}' = a\hat{i}\hat{i}' - A\hat{i}\hat{i}'$$

and inserting all of the components of $A = a\hat{i}\hat{i}' + b\hat{i}\hat{j}' + c\hat{i}\hat{k}' + d\hat{j}\hat{i}' + e\hat{j}\hat{j}' + f\hat{j}\hat{k}' + g\hat{k}\hat{i}' + h\hat{k}\hat{j}' + i\hat{k}\hat{k}'$

$$\begin{aligned} (a - A)\hat{i}\hat{i}' &= a\hat{i}\hat{i}' - A\hat{i}\hat{i}' \\ &= a\hat{i}\hat{i}' - (a\hat{i}\hat{i}' + b\hat{i}\hat{j}' + c\hat{i}\hat{k}' + d\hat{j}\hat{i}' + e\hat{j}\hat{j}' + f\hat{j}\hat{k}' + g\hat{k}\hat{i}' + h\hat{k}\hat{j}' + i\hat{k}\hat{k}')\hat{i}\hat{i}' \\ &= a\hat{i}\hat{i}' - (a\hat{i}\hat{i}'\hat{i}\hat{i}' + b\hat{i}\hat{j}'\hat{i}\hat{i}' + c\hat{i}\hat{k}'\hat{i}\hat{i}' + d\hat{j}\hat{i}'\hat{i}\hat{i}' + e\hat{j}\hat{j}'\hat{i}\hat{i}' + f\hat{j}\hat{k}'\hat{i}\hat{i}' + g\hat{k}\hat{i}'\hat{i}\hat{i}' + h\hat{k}\hat{j}'\hat{i}\hat{i}' + i\hat{k}\hat{k}'\hat{i}\hat{i}') \\ &= a\hat{i}\hat{i}' - (a\hat{i}(\hat{i}\hat{i}')\hat{i}' + b\hat{i}(\hat{j}\hat{i}')\hat{i}' + c\hat{i}(\hat{k}\hat{i}')\hat{i}' + d\hat{j}(\hat{i}\hat{i}')\hat{i}' + e\hat{j}(\hat{j}\hat{i}')\hat{i}' + f\hat{j}(\hat{k}\hat{i}')\hat{i}' + g\hat{k}(\hat{i}\hat{i}')\hat{i}' + h\hat{k}(\hat{j}\hat{i}')\hat{i}' + i\hat{k}(\hat{k}\hat{i}')\hat{i}') \\ &= a\hat{i}\hat{i}' - (a\hat{i}(1)\hat{i}' + b\hat{i}(0)\hat{i}' + c\hat{i}(0)\hat{i}' + d\hat{j}(1)\hat{i}' + e\hat{j}(0)\hat{i}' + f\hat{j}(0)\hat{i}' + g\hat{k}(1)\hat{i}' + h\hat{k}(0)\hat{i}' + i\hat{k}(0)\hat{i}') \\ &= a\hat{i}\hat{i}' - (a\hat{i}\hat{i}' + d\hat{j}\hat{i}' + g\hat{k}\hat{i}') \\ &= 0\hat{i}\hat{i}' - d\hat{j}\hat{i}' - g\hat{k}\hat{i}' \end{aligned}$$

since $\hat{i}\hat{i}' = 1$ and $\hat{j}\hat{i}' = \hat{k}\hat{i}' = \hat{j}\hat{j}' = \hat{k}\hat{j}' = \hat{j}\hat{k}' = \hat{k}\hat{k}' = 0$. Similar basis calculations for $(e - A)\hat{j}\hat{j}'$ and $(i - A)\hat{k}\hat{k}'$ result in similar results which are

$$(e - A) \hat{j} \hat{j}' = -b \hat{i} \hat{j}' + 0 \hat{j} \hat{j}' - h \hat{k} \hat{j}'$$

$$(i - A) \hat{k} \hat{k}' = -c \hat{i} \hat{k}' - f \hat{j} \hat{k}' + 0 \hat{k} \hat{k}'$$

Inserting these into the diadically expanded matrix (with the matrix nested within itself)

$$\begin{aligned} & (a - A) \hat{i} \hat{i}' + b \hat{i} \hat{j}' + c \hat{i} \hat{k}' + d \hat{j} \hat{i}' + (e - A) \hat{j} \hat{j}' + f \hat{j} \hat{k}' + g \hat{k} \hat{i}' + h \hat{k} \hat{j}' + (i - A) \hat{k} \hat{k}' \\ &= (0 \hat{i} \hat{i}' - d \hat{j} \hat{i}' - g \hat{k} \hat{i}') + b \hat{i} \hat{j}' + c \hat{i} \hat{k}' + d \hat{j} \hat{i}' + (-b \hat{i} \hat{j}' + 0 \hat{j} \hat{j}' - h \hat{k} \hat{j}') + f \hat{j} \hat{k}' + g \hat{k} \hat{i}' + h \hat{k} \hat{j}' + (-c \hat{i} \hat{k}' - f \hat{j} \hat{k}' + 0 \hat{k} \hat{k}') \\ &= 0 \hat{i} \hat{i}' + 0 \hat{i} \hat{j}' + 0 \hat{i} \hat{k}' + 0 \hat{j} \hat{i}' + 0 \hat{j} \hat{j}' + 0 \hat{j} \hat{k}' + 0 \hat{k} \hat{i}' + 0 \hat{k} \hat{j}' + 0 \hat{k} \hat{k}' \end{aligned}$$

so that $(a - A) \hat{i} \hat{i}' + b \hat{i} \hat{j}' + c \hat{i} \hat{k}' + d \hat{j} \hat{i}' + (e - A) \hat{j} \hat{j}' + f \hat{j} \hat{k}' + g \hat{k} \hat{i}' + h \hat{k} \hat{j}' + (i - A) \hat{k} \hat{k}'$ is the zero matrix, the determinant for which is also zero. **End of proof.**

Seems kind of trivial that $A - I A = 0 I$ and hence also that $\phi(A) = \det(A - I A) = \det(0 I) = 0$ since one could just subtract all elements for the zero matrix (subtracting A from itself), although the method of the above proof shows the basis mechanics of exactly what is going on (which is essentially the diad maths of multiplication by I) as a result of A being nested within itself along the diagonal. It also gives additional details as to how the two vector bases (the column and vector bases) work together as a matrix basis. It also shows how a nested basis, a basis within a basis, works. This is relevant to Hermitian matrices which have complex conjugate components which are now known to be the basis curls [basis cross products, k -cross, j -cross, and i -cross] so that these very important matrices can be converted to matrices with all real components. The basis-crosses will rotate the positions of the various components. It will also be important to recognise that there are three complex numbers [basis operators not numbers], k -cross, j -cross, and i -cross, rather than just one (i) in “complex” matrices such as the ‘spin matrices’ of quantum mechanics.

Here is a theorem in-and-of-itself, which also shows the particulars of the Cayley-Hamilton theorem. It involves the minimum polynomial which corresponds with the **minimal polynomial basis factorisation of a matrix** (developed here, with basis crosses [complex numbers], in this document – without recognising that k -cross is the complex i [which is the point of this document] this would not have been possible).

Considering the matrix

$$A = \begin{pmatrix} b & -a & 0 \\ a & b & 0 \\ 0 & 0 & b \end{pmatrix}$$

which is the diadic expansion of $b + a \hat{k} \times$ which is hence the **minimal polynomial basis factorisation** for this matrix. The two are equivalent. There are many ways that this matrix can be row reduced, as set out and shown with the introductory example. But there is only one way of doing this of lowest degree in the basis cross products [complex numbers] for the factorisation. This is the basis-curl row-reducing polynomial and is, as discussed previously, the inverse of the minimal polynomial basis factorisation $(b + a \hat{k} \times)^{-1}$.

Now consider the characteristic polynomial $\phi(\lambda)$ This is

$$\phi(\lambda) = \det(A - \lambda I) = \det \begin{pmatrix} b - \lambda & -a & 0 \\ a & b - \lambda & 0 \\ 0 & 0 & b - \lambda \end{pmatrix} = -(\lambda - b)((\lambda - b)^2 + a^2) = -(\lambda - b)(\lambda - b + a\hat{k} \times)(\lambda - b - a\hat{k} \times)$$

which hence has the characteristic [latent] roots (eigenvalues) of b , $b - a\hat{k} \times$, and $b + a\hat{k} \times$ the last being the **minimal polynomial basis factorisation for this matrix**. By the Cayley-Hamilton theorem, the matrix satisfies its own characteristic equation but there is some polynomial of smaller degree composed of some subset of these roots that the matrix also satisfies. This polynomial is the minimal polynomial and corresponds to that of the minimal polynomial basis factorisation, since this is equivalent to the matrix itself and will therefore always be a zero root (or set of roots) of the characteristic equation

$$\begin{aligned} \phi(A) &= -(A - b)(A - b + a\hat{k} \times)(A - b - a\hat{k} \times) \\ &= -(b + a\hat{k} \times - b)(b + a\hat{k} \times - b + a\hat{k} \times)(b + a\hat{k} \times - b - a\hat{k} \times) \quad \text{or} \\ &= (a\hat{k} \times)(2a\hat{k} \times)(0) \\ &= 0 \end{aligned} \qquad \begin{aligned} \phi(A) &= -(A - b)(A - b + a\hat{k} \times)(A - A) \\ &= -(A - b)(A - b + a\hat{k} \times)(0) \\ &= 0 \end{aligned}$$

and hence a minimal polynomial for this matrix is then $\lambda - b - a\hat{k} \times$. Therefore, the minimal polynomial of the Cayley-Hamilton theorem is a way to obtain the **matrix minimal polynomial basis factorisation** [once one knows how to work with **k-cross**, **j-cross**, and **i-cross** for transfer functions and resolving diffusion equations]. This would then probably be now, as from this point in time forward, the most important application of this theorem.

Quite pleased with this result actually. There are many more details to this, particularly with regard to eigenvalues, which are implicitly higher than first order and there are long chains of factors. I will be working out all of the details when I write the document that introduces $D = d / dt$ into the workings of this for dynamics.

The Cayley-Hamilton theorem is used to find powers of a matrix. If a matrix has been basis factorized into a **minimum basis-cross polynomial** then this polynomial, which is equivalent to the matrix itself, can be raised to powers and expanded as a convenient means to find a matrix power. If this polynomial is first or second order then this is done easily with binomial powers or products of binomial powers and then substitution of the relevant powers of **k-cross**, **j-cross**, and/or **i-cross** which are simple matrices of ones, zeroes, and negative ones. For $b + a\hat{k} \times$ this is particularly simple (nice maths too). For example, the cube of $A = b + a\hat{k} \times$

$$\begin{aligned} (b + a\hat{k} \times)^3 &= b^3 + 3b^2a\hat{k} \times + 3ba^2(\hat{k} \times)^2 + a^3(\hat{k} \times)^3 \\ &= b^3 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + 3b^2a \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + 3ba^2 \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + a^3 \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} b^3 & 0 & 0 \\ 0 & b^3 & 0 \\ 0 & 0 & b^3 \end{pmatrix} + \begin{pmatrix} 0 & -3b^2a & 0 \\ 3b^2a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} -3ba^2 & 0 & 0 \\ 0 & -3ba^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & a^3 & 0 \\ -a^3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} b^3 - 3ba^2 & -3b^2a + a^3 & 0 \\ 3b^2a - a^3 & b^3 - 3ba^2 & 0 \\ 0 & 0 & b^3 \end{pmatrix} \end{aligned}$$

which is hence the cubic power of the matrix A as per the expansion in powers of k -cross of the polynomial basis factorisation of A [that is, an expansion in powers of the R^3 complex number $\sqrt{-1}$ for the xy -plane diadically expanded into the matrix basis].

In this example, the matrix itself is the dyadic expansion of the principal root of the minimum polynomial. Hence the difference between the two, the matrix and diadically expanded minimum polynomial is the diadically expanded characteristic function with $\lambda = A$ and is hence identically zero. The characteristic root being multiplied by the identity puts into effect the dyadic expansion of $A = b + a \hat{k} \times$. This is therefore why the Cayley-Hamilton Theorem holds.

Theorem: The Cayley-Hamilton Theorem in R^3 holds because the matrix A is the diadic expansion of the minimum basis factorization of A , which itself is the principal root of the characteristic function and hence the root of the minimum polynomial.

Theorem: For $R^n \dots$ I also have this sorted but is going to take another equivalent large amount of new theory which I have yet to type up. nb; I have also proved that these are not actually dimensions higher than 3... it is still an R^3 vector volume.

Example: As another relevant and interesting example for considering the Cayley-Hamilton theorem and minimum polynomial in R^3 , considering the k -hat basis exponential **expk** as the rotation matrix

$$A = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The characteristic polynomial is

$$\begin{aligned} \phi(\lambda) &= \det(A - \lambda I) = \det \begin{pmatrix} \cos \theta - \lambda & -\sin \theta & 0 \\ \sin \theta & \cos \theta - \lambda & 0 \\ 0 & 0 & 1 - \lambda \end{pmatrix} = ((\cos \theta - \lambda)^2 + \sin^2 \theta)(1 - \lambda) = ((\cos \theta - \lambda)^2 - \sin^2 \theta (\hat{k} \times)^2)(1 - \lambda) \\ &= (\cos \theta + (\sin \theta) \hat{k} \times - \lambda)(\cos \theta - (\sin \theta) \hat{k} \times - \lambda)(1 - \lambda) \end{aligned}$$

but these sines and cosines are those for the k -hat rotation axis so this can be written as

$$\begin{aligned} \phi(\lambda) &= \det(A - \lambda I) \\ &= (\cos \theta + (\sin \theta) \hat{k} \times - \lambda)(\cos \theta - (\sin \theta) \hat{k} \times - \lambda)(1 - \lambda) \\ &= (\mathbf{expk} \theta - \lambda)(\mathbf{expk}(-\theta) - \lambda)(1 - \lambda) \end{aligned}$$

for which the characteristic roots are **expk**, the conjugate **expk**, and the unity rotation eigenvalue. Then the principal root of the minimum polynomial is **expk**, that is, a characteristic root $\cos \theta + (\sin \theta) \hat{k} \times$ that is equivalent to the matrix A itself once diadically expanded. Therefore, the Cayley-Hamilton theorem holds for R^3 because this principal root **expk** is identically equal to the matrix A and so always zeroes its own characteristic polynomial

$$\begin{aligned} \phi(A) &= \phi(\mathbf{expk} \theta) \\ &= (\mathbf{expk} \theta - \mathbf{expk} \theta)(\mathbf{expk}(-\theta) - \mathbf{expk} \theta)(I - \mathbf{expk} \theta) \\ &= (0I)(\mathbf{expk}(-\theta) - \mathbf{expk} \theta)(I - \mathbf{expk} \theta) \\ &= 0I \end{aligned}$$

Hence the matrix itself is one of the roots of the characteristic equation, the minimal polynomial being that where the extraneous roots are ignored

$$m(\lambda) = (\mathbf{expk}\theta - \lambda) \\ = (\cos k\theta + (\sin k\theta)\hat{k} \times) - \lambda$$

Diadic Mathematics – inner product, outer product, and matrix product

Here I outline some more diadic mathematics, here that of matrix multiplication. I have omitted vector-times-matrix and matrix-times-vector since those multiplications are a subset of matrix-times-matrix outlined here.

A matrix in full basis expanded form is

$$A = a\hat{i}\hat{i}' + b\hat{i}\hat{j}' + c\hat{i}\hat{k}' + d\hat{j}\hat{i}' + e\hat{j}\hat{j}' + f\hat{j}\hat{k}' + g\hat{k}\hat{i}' + h\hat{k}\hat{j}' + i\hat{k}\hat{k}' \quad \text{i.e., from} \quad \begin{pmatrix} a(\hat{i}\hat{i}') & b(\hat{i}\hat{j}') & c(\hat{i}\hat{k}') \\ d(\hat{j}\hat{i}') & e(\hat{j}\hat{j}') & f(\hat{j}\hat{k}') \\ g(\hat{k}\hat{i}') & h(\hat{k}\hat{j}') & i(\hat{k}\hat{k}') \end{pmatrix}$$

That same basis factorised into **row basis expanded form** is

$$A = (a\hat{i}\hat{i}' + b\hat{i}\hat{j}' + c\hat{i}\hat{k}') + (d\hat{j}\hat{i}' + e\hat{j}\hat{j}' + f\hat{j}\hat{k}') + (g\hat{k}\hat{i}' + h\hat{k}\hat{j}' + i\hat{k}\hat{k}') \\ = \hat{i}(a\hat{i}' + b\hat{j}' + c\hat{k}') + \hat{j}(d\hat{i}' + e\hat{j}' + f\hat{k}') + \hat{k}(g\hat{i}' + h\hat{j}' + i\hat{k}')$$

or factorised in column basis expanded form

$$A = (a\hat{i}\hat{i}' + d\hat{j}\hat{i}' + g\hat{k}\hat{i}') + (b\hat{i}\hat{j}' + e\hat{j}\hat{j}' + h\hat{k}\hat{j}') + (c\hat{i}\hat{k}' + f\hat{j}\hat{k}' + i\hat{k}\hat{k}') \\ = (a\hat{i} + d\hat{j} + g\hat{k})\hat{i}' + (b\hat{i} + e\hat{j} + h\hat{k})\hat{j}' + (c\hat{i} + f\hat{j} + i\hat{k})\hat{k}'$$

As how mathematics has defined it, matrix multiplication multiplies the rows of the first matrix with the columns of the second matrix to generate the matrix elements of the resulting matrix product. Another matrix B in **column basis expanded form** is

$$B = (\alpha\hat{i}\hat{i}' + \delta\hat{j}\hat{i}' + \gamma\hat{k}\hat{i}') + (\beta\hat{i}\hat{j}' + \varepsilon\hat{j}\hat{j}' + \eta\hat{k}\hat{j}') + (\chi\hat{i}\hat{k}' + \phi\hat{j}\hat{k}' + \iota\hat{k}\hat{k}') \\ = (\alpha\hat{i} + \delta\hat{j} + \gamma\hat{k})\hat{i}' + (\beta\hat{i} + \varepsilon\hat{j} + \eta\hat{k})\hat{j}' + (\chi\hat{i}\hat{k}' + \phi\hat{j}\hat{k}' + \iota\hat{k})\hat{k}'$$

Multiplying any factored form (row factored or column factored) by any other factored form (row factored or column factored) will work (for the matrix-matrix multiplication), since the factored forms are equivalent (since it is just rearranged basis algebra); but which combination of factorizations is easiest? I had thought the following method would work easiest but turned out to involve substantial algebra. The other expanded forms may provide prove easier, but from the following algebra and given that they are equivalent, it probably all works out about the same in terms of lines of working. Multiplying A to B results in 3 factored terms times another 3 factored terms giving the 9 scalar terms tied to the relevant matrix basis terms

$$\begin{aligned}
AB &= \left((a\hat{i} + d\hat{j} + g\hat{k})\hat{i}' + (b\hat{i} + e\hat{j} + h\hat{k})\hat{j}' + (c\hat{i} + f\hat{j} + i\hat{k})\hat{k}' \right) \left((\alpha\hat{i} + \delta\hat{j} + \gamma\hat{k})\hat{i}' + (\beta\hat{i} + \varepsilon\hat{j} + \eta\hat{k})\hat{j}' + (\chi\hat{i} + \phi\hat{j} + \iota\hat{k})\hat{k}' \right) \\
&= (a\hat{i} + d\hat{j} + g\hat{k})\hat{i}' \left((\alpha\hat{i} + \delta\hat{j} + \gamma\hat{k})\hat{i}' + (\beta\hat{i} + \varepsilon\hat{j} + \eta\hat{k})\hat{j}' + (\chi\hat{i} + \phi\hat{j} + \iota\hat{k})\hat{k}' \right) \\
&\quad + (b\hat{i} + e\hat{j} + h\hat{k})\hat{j}' \left((\alpha\hat{i} + \delta\hat{j} + \gamma\hat{k})\hat{i}' + (\beta\hat{i} + \varepsilon\hat{j} + \eta\hat{k})\hat{j}' + (\chi\hat{i} + \phi\hat{j} + \iota\hat{k})\hat{k}' \right) \\
&\quad + (c\hat{i} + f\hat{j} + i\hat{k})\hat{k}' \left((\alpha\hat{i} + \delta\hat{j} + \gamma\hat{k})\hat{i}' + (\beta\hat{i} + \varepsilon\hat{j} + \eta\hat{k})\hat{j}' + (\chi\hat{i} + \phi\hat{j} + \iota\hat{k})\hat{k}' \right) \\
&= (a\hat{i} + d\hat{j} + g\hat{k}) \left((\alpha\hat{i}'\hat{i} + \delta\hat{i}'\hat{j} + \gamma\hat{i}'\hat{k})\hat{i}' + (\beta\hat{i}'\hat{i} + \varepsilon\hat{i}'\hat{j} + \eta\hat{i}'\hat{k})\hat{j}' + (\chi\hat{i}'\hat{i} + \phi\hat{i}'\hat{j} + \iota\hat{i}'\hat{k})\hat{k}' \right) \\
&\quad + (b\hat{i} + e\hat{j} + h\hat{k}) \left((\alpha\hat{j}'\hat{i} + \delta\hat{j}'\hat{j} + \gamma\hat{j}'\hat{k})\hat{i}' + (\beta\hat{j}'\hat{i} + \varepsilon\hat{j}'\hat{j} + \eta\hat{j}'\hat{k})\hat{j}' + (\chi\hat{j}'\hat{i} + \phi\hat{j}'\hat{j} + \iota\hat{j}'\hat{k})\hat{k}' \right) \\
&\quad + (c\hat{i} + f\hat{j} + i\hat{k}) \left((\alpha\hat{k}'\hat{i} + \delta\hat{k}'\hat{j} + \gamma\hat{k}'\hat{k})\hat{i}' + (\beta\hat{k}'\hat{i} + \varepsilon\hat{k}'\hat{j} + \eta\hat{k}'\hat{k})\hat{j}' + (\chi\hat{k}'\hat{i} + \phi\hat{k}'\hat{j} + \iota\hat{k}'\hat{k})\hat{k}' \right)
\end{aligned}$$

since $\hat{j}'\hat{i} = \hat{k}'\hat{i} = \hat{i}'\hat{j} = \hat{k}'\hat{j} = \hat{j}'\hat{k} = \hat{i}'\hat{k} = 0$. Since also $\hat{i}'\hat{i} = \hat{j}'\hat{j} = \hat{k}'\hat{k} = 1$

$$\begin{aligned}
AB &= (a\hat{i} + d\hat{j} + g\hat{k}) \left((\alpha\hat{i}'\hat{i})\hat{i}' + (\beta\hat{i}'\hat{i})\hat{j}' + (\chi\hat{i}'\hat{i})\hat{k}' \right) \\
&\quad + (b\hat{i} + e\hat{j} + h\hat{k}) \left((\delta\hat{j}'\hat{j})\hat{i}' + (\varepsilon\hat{j}'\hat{j})\hat{j}' + (\phi\hat{j}'\hat{j})\hat{k}' \right) \\
&\quad + (c\hat{i} + f\hat{j} + i\hat{k}) \left((\gamma\hat{k}'\hat{k})\hat{i}' + (\eta\hat{k}'\hat{k})\hat{j}' + (\iota\hat{k}'\hat{k})\hat{k}' \right) \\
&= (a\hat{i} + d\hat{j} + g\hat{k}) \left((\alpha)\hat{i}' + (\beta)\hat{j}' + (\chi)\hat{k}' \right) + (b\hat{i} + e\hat{j} + h\hat{k}) \left((\delta)\hat{i}' + (\varepsilon)\hat{j}' + (\phi)\hat{k}' \right) + (c\hat{i} + f\hat{j} + i\hat{k}) \left((\gamma)\hat{i}' + (\eta)\hat{j}' + (\iota)\hat{k}' \right) \\
&= \alpha(a\hat{i} + d\hat{j} + g\hat{k})\hat{i}' + \beta(a\hat{i} + d\hat{j} + g\hat{k})\hat{j}' + \chi(a\hat{i} + d\hat{j} + g\hat{k})\hat{k}' \\
&\quad + \delta(b\hat{i} + e\hat{j} + h\hat{k})\hat{i}' + \varepsilon(b\hat{i} + e\hat{j} + h\hat{k})\hat{j}' + \phi(b\hat{i} + e\hat{j} + h\hat{k})\hat{k}' \\
&\quad + \gamma(c\hat{i} + f\hat{j} + i\hat{k})\hat{i}' + \eta(c\hat{i} + f\hat{j} + i\hat{k})\hat{j}' + \iota(c\hat{i} + f\hat{j} + i\hat{k})\hat{k}'
\end{aligned}$$

which is in **column basis expanded form** so now just needs rearranging

$$\begin{aligned}
AB &= (a\alpha\hat{i} + d\alpha\hat{j} + g\alpha\hat{k})\hat{i}' + (a\beta\hat{i} + d\beta\hat{j} + g\beta\hat{k})\hat{j}' + (a\chi\hat{i} + d\chi\hat{j} + g\chi\hat{k})\hat{k}' \\
&\quad + (b\delta\hat{i} + e\delta\hat{j} + h\delta\hat{k})\hat{i}' + (b\varepsilon\hat{i} + e\varepsilon\hat{j} + h\varepsilon\hat{k})\hat{j}' + (b\phi\hat{i} + e\phi\hat{j} + h\phi\hat{k})\hat{k}' \\
&\quad + (c\gamma\hat{i} + f\gamma\hat{j} + i\gamma\hat{k})\hat{i}' + (c\eta\hat{i} + f\eta\hat{j} + i\eta\hat{k})\hat{j}' + (c\iota\hat{i} + f\iota\hat{j} + i\iota\hat{k})\hat{k}' \\
&= (a\alpha\hat{i} + d\alpha\hat{j} + g\alpha\hat{k} + b\delta\hat{i} + e\delta\hat{j} + h\delta\hat{k} + c\gamma\hat{i} + f\gamma\hat{j} + i\gamma\hat{k})\hat{i}' \\
&\quad + (a\beta\hat{i} + d\beta\hat{j} + g\beta\hat{k} + b\varepsilon\hat{i} + e\varepsilon\hat{j} + h\varepsilon\hat{k} + c\eta\hat{i} + f\eta\hat{j} + i\eta\hat{k})\hat{j}' \\
&\quad + (a\chi\hat{i} + d\chi\hat{j} + g\chi\hat{k} + b\phi\hat{i} + e\phi\hat{j} + h\phi\hat{k} + c\iota\hat{i} + f\iota\hat{j} + i\iota\hat{k})\hat{k}' \\
&= (a\alpha + b\delta + c\gamma)\hat{i}'\hat{i}' + (d\alpha + e\delta + f\gamma)\hat{j}'\hat{i}' + (g\alpha + h\delta + i\gamma)\hat{k}'\hat{i}' \\
&\quad + (a\beta + b\varepsilon + c\eta)\hat{i}'\hat{j}' + (d\beta + e\varepsilon + f\eta)\hat{j}'\hat{j}' + (g\beta + h\varepsilon + i\eta)\hat{k}'\hat{j}' \\
&\quad + (a\chi + b\phi + c\iota)\hat{i}'\hat{k}' + (d\chi + e\phi + f\iota)\hat{j}'\hat{k}' + (g\chi + h\phi + i\iota)\hat{k}'\hat{k}'
\end{aligned}$$

which is now in matrix form with the 9 matrix basis components $\hat{i}'\hat{i}', \hat{j}'\hat{i}', \hat{k}'\hat{i}', \hat{i}'\hat{j}', \hat{j}'\hat{j}', \hat{k}'\hat{j}', \hat{i}'\hat{k}', \hat{j}'\hat{k}',$ and $\hat{k}'\hat{k}'$. In standard format, therefore

$$AB = \begin{pmatrix} a\alpha + b\delta + c\gamma & a\beta + b\varepsilon + c\eta & a\chi + b\phi + c\iota \\ d\alpha + e\delta + f\gamma & d\beta + e\varepsilon + f\eta & d\chi + e\phi + f\iota \\ g\alpha + h\delta + i\gamma & g\beta + h\varepsilon + i\eta & g\chi + h\phi + i\iota \end{pmatrix}$$

which is the matrix with the components of the row components of A times the column components of B demonstrating the mechanism of matrix multiplication, as required. What was the point of the above basis algebra of matrix multiplication? I was wanting to demonstrate that linear algebra is a composition of complex numbers. I was also wanting to show that when a basis system is considered as a set of complex numbers it becomes intuitive known-to-be-good mathematics consistent with that in complex algebra and calculus (with additional methods, including for example, 3by1 column vector by 3by3 multiplications consistent with the complex number properties on the column vector within the matrix basis). The practical reason here though, in addition to purely mathematical interest, was because I need to know how it all works (down to the very last basis component and trivial detail) so that it can hopefully be generalized it into a full Rubiks ($3 \times 3 \times 3 = 27$ term quaternion. I am hoping that despite there being a very large number of terms, once it is worked out it will become **(1)** easy to apply, **(2)** will all be intuitive, and **(3)** will end up working for the Rubiks cube as single multiplications of **expk**, **expj**, and **expi**. (for the rotations of the 3×3 slices of the Rubiks cube)... and so the majority of terms can be latent.

Contrasting singular k-cross with the $\pi/2$ k-hat exponentials for the quartic roots

The discussion and mathematics in this section regards contrasting k -cross, j -cross, and i -cross with **expk**($\pi/2$), **expj**($\pi/2$), and **expi**($\pi/2$) as the full rank quarter circle rotations (those for a Rubiks cube). The operators **expk**($\pi/2$), **expj**($\pi/2$), and **expi**($\pi/2$) are the same as k -cross, j -cross, and i -cross except with the additional rotational eigenvalue. The additional eigenvalue is the single matrix basis-component added to the diagonal to keep the rotation positive definite. So for k -cross the unity rotation eigenvalue is kk' , for j -cross this is jj' , and for i -cross this is ii' . When these operate on the components of a **position vector** or **colour vector**, the components of the same basis component convert to the vector basis component (k -hat for kk' , for example, multiply to give/leave k -hat) leaving that component unchanged whilst those of the different basis component zero (j -hat or i -hat for kk' , for example, null multiply) those components which means that they are not double counted with the permuted components.

As applicable to **matrices** as outer products of **columns** and **rows**:

- (1)** unprimed cross [which is a column cross product] will cross out a row and swap the other two rows (applied from the left)
(2) primed cross [which is a row cross product] will cross out a column and swap the other two columns (applied from the right)

$$\hat{k} \times \begin{pmatrix} a \hat{i} \hat{i}' & b \hat{i} \hat{j}' & c \hat{i} \hat{k}' \\ d \hat{j} \hat{i}' & e \hat{j} \hat{j}' & f \hat{j} \hat{k}' \\ g \hat{k} \hat{i}' & h \hat{k} \hat{j}' & i \hat{k} \hat{k}' \end{pmatrix} = \begin{pmatrix} a \hat{k} \times \hat{i} \hat{i}' & b \hat{k} \times \hat{i} \hat{j}' & c \hat{k} \times \hat{i} \hat{k}' \\ d \hat{k} \times \hat{j} \hat{i}' & e \hat{k} \times \hat{j} \hat{j}' & f \hat{k} \times \hat{j} \hat{k}' \\ g \hat{k} \times \hat{k} \hat{i}' & h \hat{k} \times \hat{k} \hat{j}' & i \hat{k} \times \hat{k} \hat{k}' \end{pmatrix} = \begin{pmatrix} d \hat{i} \hat{i}' & e \hat{i} \hat{j}' & f \hat{i} \hat{k}' \\ -a \hat{j} \hat{i}' & -b \hat{j} \hat{j}' & -c \hat{j} \hat{k}' \\ 0 \hat{k} \hat{i}' & 0 \hat{k} \hat{j}' & 0 \hat{k} \hat{k}' \end{pmatrix} \quad (\text{zeroed third row})$$

and

$$\begin{pmatrix} a \hat{i} \hat{i}' & b \hat{i} \hat{j}' & c \hat{i} \hat{k}' \\ d \hat{j} \hat{i}' & e \hat{j} \hat{j}' & f \hat{j} \hat{k}' \\ g \hat{k} \hat{i}' & h \hat{k} \hat{j}' & i \hat{k} \hat{k}' \end{pmatrix} \times \hat{k}' = \begin{pmatrix} a \hat{i} \hat{i}' \times \hat{k}' & b \hat{i} \hat{j}' \times \hat{k}' & c \hat{i} \hat{k}' \times \hat{k}' \\ d \hat{j} \hat{i}' \times \hat{k}' & e \hat{j} \hat{j}' \times \hat{k}' & f \hat{j} \hat{k}' \times \hat{k}' \\ g \hat{k} \hat{i}' \times \hat{k}' & h \hat{k} \hat{j}' \times \hat{k}' & i \hat{k} \hat{k}' \times \hat{k}' \end{pmatrix} = \begin{pmatrix} b \hat{i} \hat{i}' & -a \hat{i} \hat{j}' & 0 \hat{i} \hat{k}' \\ e \hat{j} \hat{i}' & -d \hat{j} \hat{j}' & 0 \hat{j} \hat{k}' \\ h \hat{k} \hat{i}' & -g \hat{k} \hat{j}' & 0 \hat{k} \hat{k}' \end{pmatrix} \quad (\text{zeroed third column})$$

The k -cross operators with the eigenvalues (the k -hat exponentials at $\pi/2$), $\hat{k}\hat{k}' + \hat{k} \times$ and $\hat{k}\hat{k}' + \times \hat{k}'$, work similarly but with the difference that the k -hat row or k -hat column are not crossed out (not zeroed) and rather left as is

$$(\hat{k}\hat{k}' + \hat{k} \times) \begin{pmatrix} a \hat{i} \hat{i}' & b \hat{i} \hat{j}' & c \hat{i} \hat{k}' \\ d \hat{j} \hat{i}' & e \hat{j} \hat{j}' & f \hat{j} \hat{k}' \\ g \hat{k} \hat{i}' & h \hat{k} \hat{j}' & i \hat{k} \hat{k}' \end{pmatrix} = \begin{pmatrix} d \hat{i} \hat{i}' & e \hat{i} \hat{j}' & f \hat{i} \hat{k}' \\ -a \hat{j} \hat{i}' & -b \hat{j} \hat{j}' & -c \hat{j} \hat{k}' \\ g \hat{k} \hat{i}' & h \hat{k} \hat{j}' & i \hat{k} \hat{k}' \end{pmatrix} \quad (\text{non-zeroed third row})$$

and

$$\begin{pmatrix} a \hat{i} \hat{i}' & b \hat{i} \hat{j}' & c \hat{i} \hat{k}' \\ d \hat{j} \hat{i}' & e \hat{j} \hat{j}' & f \hat{j} \hat{k}' \\ g \hat{k} \hat{i}' & h \hat{k} \hat{j}' & i \hat{k} \hat{k}' \end{pmatrix} (\hat{k}\hat{k}' + \times \hat{k}') = \begin{pmatrix} b \hat{i} \hat{i}' & -a \hat{i} \hat{j}' & c \hat{i} \hat{k}' \\ e \hat{j} \hat{i}' & -d \hat{j} \hat{j}' & f \hat{j} \hat{k}' \\ h \hat{k} \hat{i}' & -g \hat{k} \hat{j}' & i \hat{k} \hat{k}' \end{pmatrix} \quad (\text{non-zeroed third column})$$

and this is the difference (kk') between the complex number (k -cross) and the complex exponential at $\pi/2$ (k -hat exponential at $\pi/2$). This has relevance to differential systems and the extension to Rubiks quaternions which I will be attempting to complete the theory in the near future.

Further Basis notation for $\pi/2$ rotations

Writing the matrix in matrix form whilst latently showing the matrix basis components in their respective locations within the standard vector and matrix formats, which here is meant to signify that the basis components are behind the scenes (but made visible here) and (otherwise) invisibly and latently generating the space for the vectors and matrices

$$\begin{pmatrix} a \hat{i} \hat{i}' & b \hat{i} \hat{j}' & c \hat{i} \hat{k}' \\ d \hat{j} \hat{i}' & e \hat{j} \hat{j}' & f \hat{j} \hat{k}' \\ g \hat{k} \hat{i}' & h \hat{k} \hat{j}' & i \hat{k} \hat{k}' \end{pmatrix} \quad \begin{pmatrix} a \hat{i}' & b \hat{j}' & c \hat{k}' \end{pmatrix} \quad \begin{pmatrix} a \hat{i} \\ b \hat{j} \\ c \hat{k} \end{pmatrix}$$

The quarter turn rotation matrices ($\pi/2$ k -hat, $\pi/2$ j -hat, and $\pi/2$ i -hat exponentials) can be written in terms of the matrix basis components

$$e^{\frac{\pi}{2} \hat{k} \times} (\cdot) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} (\cdot) = e^{\frac{\pi}{2} \hat{k} \times} = \begin{pmatrix} 0(\hat{i} \hat{i}') & -1(\hat{i} \hat{j}') & 0(\hat{i} \hat{k}') \\ 1(\hat{j} \hat{i}') & 0(\hat{j} \hat{j}') & 0(\hat{j} \hat{k}') \\ 0(\hat{k} \hat{i}') & 0(\hat{k} \hat{j}') & 1(\hat{k} \hat{k}') \end{pmatrix} = -1(\hat{i} \hat{j}') + 1(\hat{j} \hat{i}') + 1(\hat{k} \hat{k}') = -\hat{i} \hat{j}' + \hat{j} \hat{i}' + \hat{k} \hat{k}'$$

$$e^{\frac{\pi}{2} \hat{j} \times} (\cdot) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} (\cdot) = e^{\frac{\pi}{2} \hat{j} \times} = \begin{pmatrix} 0(\hat{i} \hat{i}') & 0(\hat{i} \hat{j}') & 1(\hat{i} \hat{k}') \\ 1(\hat{j} \hat{i}') & 1(\hat{j} \hat{j}') & 0(\hat{j} \hat{k}') \\ -1(\hat{k} \hat{i}') & 0(\hat{k} \hat{j}') & 0(\hat{k} \hat{k}') \end{pmatrix} = 1(\hat{i} \hat{k}') + 1(\hat{j} \hat{j}') - 1(\hat{k} \hat{i}') = \hat{i} \hat{k}' + \hat{j} \hat{j}' - \hat{k} \hat{i}'$$

$$e^{\frac{\pi}{2} \hat{i} \times} (\cdot) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} (\cdot) = e^{\frac{\pi}{2} \hat{i} \times} = \begin{pmatrix} 1(\hat{i} \hat{i}') & 0(\hat{i} \hat{j}') & 0(\hat{i} \hat{k}') \\ 0(\hat{j} \hat{i}') & 0(\hat{j} \hat{j}') & -1(\hat{j} \hat{k}') \\ 0(\hat{k} \hat{i}') & 1(\hat{k} \hat{j}') & 0(\hat{k} \hat{k}') \end{pmatrix} = 1(\hat{i} \hat{i}') - 1(\hat{j} \hat{k}') + 1(\hat{k} \hat{j}') = \hat{i} \hat{i}' - \hat{j} \hat{k}' + \hat{k} \hat{j}'$$

i.e., the quarter turn rotation matrices ($\pi/2$ $\hat{k}$ -hat, $\pi/2$ $\hat{j}$ -hat, and $\pi/2$ $\hat{i}$ -hat exponentials; **expk**, **expj**, and **expi** at $\pi/2$) can be written succinctly as

$$e^{\frac{\pi}{2} \hat{k} \times} = -\hat{i} \hat{j}' + \hat{j} \hat{i}' + \hat{k} \hat{k}' \quad e^{\frac{\pi}{2} \hat{j} \times} = \hat{i} \hat{k}' + \hat{j} \hat{j}' - \hat{k} \hat{i}' \quad e^{\frac{\pi}{2} \hat{i} \times} = \hat{i} \hat{i}' - \hat{j} \hat{k}' + \hat{k} \hat{j}'$$

The conjugate of these three operators (for the corresponding three conjugate operators) changes the sign of the mixed terms [the two terms other than the rotation eigenvalue term] such as the matrix basis component of $\hat{i}$ -hat with primed $\hat{j}$ -hat for example. These are the terms of the basis cross product that are permuted. The sign of the unitary rotational eigenvalue [the remaining term] remains unchanged. The three conjugates are

$$e^{-\frac{\pi}{2} \hat{k} \times} = \hat{i} \hat{j}' - \hat{j} \hat{i}' + \hat{k} \hat{k}' \quad e^{-\frac{\pi}{2} \hat{j} \times} = -\hat{i} \hat{k}' + \hat{j} \hat{j}' + \hat{k} \hat{i}' \quad e^{-\frac{\pi}{2} \hat{i} \times} = \hat{i} \hat{i}' + \hat{j} \hat{k}' - \hat{k} \hat{j}'$$

Note that since the difference between a $\pi/2$ basis-hat exponential ($e^{\frac{\pi}{2} \hat{k} \times}$, $e^{\frac{\pi}{2} \hat{j} \times}$, and $e^{\frac{\pi}{2} \hat{i} \times}$) and the basis-cross (complex number $\hat{k} \times$, $\hat{j} \times$, and $\hat{i} \times$) is the rotational eigenvalue, the basis-hat, before diadic expansion, can also be written with dot products

$$e^{\frac{\pi}{2} \hat{k} \times} = \hat{k} \hat{k} \bullet + \hat{k} \times = \hat{k} (\hat{k} \bullet + \times) \quad e^{\frac{\pi}{2} \hat{j} \times} = \hat{j} \hat{j} \bullet + \hat{j} \times = \hat{j} (\hat{j} \bullet + \times) \quad e^{\frac{\pi}{2} \hat{i} \times} = \hat{i} \hat{i} \bullet + \hat{i} \times = \hat{i} (\hat{i} \bullet + \times)$$

where the first term is the dot-product [so that the rotational eigenvalue is retained] and the second is the cross-product [the complex number]. I am currently of the opinion in the circumstances of what I am doing here [and common among those that have lectured me] that it is better is not to use dot product but to instead use inner product so these are

$$e^{\frac{\pi}{2} \hat{k} \times} = \hat{k} \hat{k}' + \hat{k} \times = \hat{k} (\hat{k}' + \times) \quad e^{\frac{\pi}{2} \hat{j} \times} = \hat{j} \hat{j}' + \hat{j} \times = \hat{j} (\hat{j}' + \times) \quad e^{\frac{\pi}{2} \hat{i} \times} = \hat{i} \hat{i}' + \hat{i} \times = \hat{i} (\hat{i}' + \times)$$

The conjugate changes the sign of the complex term, whilst leaving the sign of the retained unitary eigenvalue unchanged, and so the conjugates are

$$e^{-\frac{\pi}{2} \hat{k} \times} = \hat{k} \hat{k}' - \hat{k} \times = \hat{k} (\hat{k}' - \times) \quad e^{-\frac{\pi}{2} \hat{j} \times} = \hat{j} \hat{j}' - \hat{j} \times = \hat{j} (\hat{j}' - \times) \quad e^{-\frac{\pi}{2} \hat{i} \times} = \hat{i} \hat{i}' - \hat{i} \times = \hat{i} (\hat{i}' - \times)$$

Hence, the six ($\pi/2$) rotational operators can be written as either

$$(1) \quad \hat{k} \hat{k}' + \hat{k} \times \quad \hat{j} \hat{j}' + \hat{j} \times \quad \hat{i} \hat{i}' + \hat{i} \times \quad \hat{k} \hat{k}' - \hat{k} \times \quad \hat{j} \hat{j}' - \hat{j} \times \quad \hat{i} \hat{i}' - \hat{i} \times$$

or

$$(2) \quad -\hat{i} \hat{j}' + \hat{j} \hat{i}' + \hat{k} \hat{k}' \quad \hat{i} \hat{k}' + \hat{j} \hat{j}' - \hat{k} \hat{i}' \quad \hat{i} \hat{i}' - \hat{j} \hat{k}' + \hat{k} \hat{j}' \quad \hat{i} \hat{j}' - \hat{j} \hat{i}' + \hat{k} \hat{k}' \quad -\hat{i} \hat{k}' + \hat{j} \hat{j}' + \hat{k} \hat{i}' \quad \hat{i} \hat{i}' + \hat{j} \hat{k}' - \hat{k} \hat{j}'$$

or as **(3)** basis-hat exponentials $e^{\frac{\pi}{2}\hat{k}\times}$, $e^{\frac{\pi}{2}\hat{j}\times}$, $e^{\frac{\pi}{2}\hat{i}\times}$, $e^{-\frac{\pi}{2}\hat{k}\times}$, $e^{-\frac{\pi}{2}\hat{j}\times}$, $e^{-\frac{\pi}{2}\hat{i}\times}$ or **(4)** in rotational matrix form. That is four main ways to express the quarter turn basis-hat (complex) exponentials. I also use shorthand notation which is $k, j, i, k', j',$ and i' [especially for cubing algorithms] which corresponds to complex notation. This makes five ways to write the six quarter-turn operators. Note that for the shorthand notation, the cross $\times$ and the rotational eigenvalue are latent but assumed.

The position vectors of the bits can be written in terms of the position vector components, noting that these are the vector components taken in **singlets** (for **faces**), **doublets** (for **edges**) and **triplets** (for **corners**), for where there are 6 **faces**, 12 **edges**, and 8 **corners**

faces: 3choose1 times (2signs)¹ = 1 times 6 = 6 faces
edges: 3choose2 times (2signs)² = 3 times 4 = 12 edges
corners: 3choose3 times (2signs)³ = 1 times 8 = 8 corners

6 faces: $\hat{i}, -\hat{i}, \hat{j}, -\hat{j}, \hat{k}, -\hat{k}$ ($\pm\hat{i}, \pm\hat{j}, \pm\hat{k}$)
12 edges: $\hat{i} + \hat{j}, -\hat{i} + \hat{j}, \hat{i} - \hat{j}, -\hat{i} - \hat{j}, \hat{i} + \hat{k}, -\hat{i} + \hat{k}, \hat{i} - \hat{k}, -\hat{i} - \hat{k}, \hat{j} + \hat{k}, -\hat{j} + \hat{k}, \hat{j} - \hat{k}, -\hat{j} - \hat{k}$
8 corners: $\hat{i} + \hat{j} + \hat{k}, \hat{i} + \hat{j} - \hat{k}, \hat{i} - \hat{j} + \hat{k}, \hat{i} - \hat{j} - \hat{k}, -\hat{i} + \hat{j} + \hat{k}, -\hat{i} + \hat{j} - \hat{k}, -\hat{i} - \hat{j} + \hat{k}, -\hat{i} - \hat{j} - \hat{k}$

Since both the rotation operators and the bit positions can be written in terms of basis components, the overall bit tracking equations can be written out explicitly in terms of basis components. This gives an alternative way, from vector and matrix notation, to set out the working. This basis notation may be better in certain circumstances since this works with the algebra of the basis systems rather than intuitively as how most, including myself, learned it by route in uni [fair to assume that most if not all whom persevere reading this have sat, and like, linear algebra]. Rows times columns, rows times columns, rows times columns, The basis notation for positions and rotations looks like a system of complex numbers, which is not coincidental. This is what I had earlier become convinced of, that the standard basis is a set of complex numbers and, some time later, that a Rubiks cube is a self-contained system of complex numbers and basis rotations. I think I am now demonstrating this quite successfully.

For the examples of identity operations, the basis notation is

center bit:

$$\begin{aligned} e^{\frac{\pi}{2}\hat{k}\times} e^{\frac{\pi}{2}\hat{i}\times} \left(e^{\frac{\pi}{2}\hat{j}\times} \right)^T \hat{i} &= (\hat{k}\hat{k}' + \hat{k}\times) (\hat{i}\hat{i}' + \hat{i}\times) (\hat{j}\hat{j}' - \hat{j}\times) \hat{i} \\ &= (-\hat{i}\hat{j}' + \hat{j}\hat{i}' + \hat{k}\hat{k}') (\hat{i}\hat{i}' - \hat{j}\hat{k}' + \hat{k}\hat{j}') (-\hat{i}\hat{k}' + \hat{j}\hat{j}' + \hat{k}\hat{i}') \hat{i} \\ &= \hat{i} \end{aligned}$$

edge bit:

$$\begin{aligned} e^{\frac{\pi}{2}\hat{i}\times} \left(e^{\frac{\pi}{2}\hat{k}\times} \right)^2 \left(e^{\frac{\pi}{2}\hat{j}\times} \right)^T \left(e^{\frac{\pi}{2}\hat{j}\times} \right)^2 (\hat{i} + \hat{j}) &= (\hat{i}\hat{i}' + \hat{i}\times) (\hat{k}\hat{k}' + \hat{k}\times)^2 (\hat{i}\hat{i}' - \hat{i}\times) (\hat{j}\hat{j}' + \hat{j}\times)^2 (\hat{i} + \hat{j}) \\ &= (\hat{i}\hat{i}' - \hat{j}\hat{k}' + \hat{k}\hat{j}') (-\hat{i}\hat{j}' + \hat{j}\hat{i}' + \hat{k}\hat{k}')^2 (\hat{i}\hat{i}' + \hat{j}\hat{k}' - \hat{k}\hat{j}') (\hat{i}\hat{k}' + \hat{j}\hat{j}' - \hat{k}\hat{i}')^2 (\hat{i} + \hat{j}) \\ &= \hat{i} + \hat{j} \end{aligned}$$

corner bit:

$$\begin{aligned}
e^{\frac{\pi}{2}\hat{k}\times} e^{\frac{\pi}{2}\hat{j}\times} \left(e^{\frac{\pi}{2}\hat{k}\times} \right)^T e^{\frac{\pi}{2}\hat{i}\times} (\hat{i} + \hat{j} + \hat{k}) &= (\hat{k}\hat{k}' + \hat{k}\times)(\hat{j}\hat{j}' + \hat{j}\times)(\hat{k}\hat{k}' - \hat{k}\times)(\hat{i}\hat{i}' + \hat{i}\times)(\hat{i} + \hat{j} + \hat{k}) \\
&= (-\hat{i}\hat{j}' + \hat{j}\hat{i}' + \hat{k}\hat{k}')(\hat{i}\hat{k}' + \hat{j}\hat{j}' - \hat{k}\hat{i}')(\hat{i}\hat{j}' - \hat{j}\hat{i}' + \hat{k}\hat{k}')(\hat{i}\hat{i}' - \hat{j}\hat{k}' + \hat{k}\hat{j}')(\hat{i} + \hat{j} + \hat{k}) \\
&= \hat{i} + \hat{j} + \hat{k}
\end{aligned}$$

or equivalently with **primed row position vectors** at the left rather than as above with **un-primed column position vectors** at the right. It looks just like complex numbers, and it is all done simply with the linear algebra standard basis and the binary operator $\times$. The way that this looks like complex numbers is beautiful to me.

Orientation tracking and Colour Vectors

The 6 colours of a Rubiks cube - red, orange, blue, green, yellow, white - each have convenient distinct first letters, that is, red **r**, orange **o**, blue **b**, green **g**, yellow **y**, and white **w**. Similarly as visually on a Rubiks cube, the colours can then act as a marker of orientation on the standard basis. These similarly act as a triplet bar code (plus one and two redundancies for faces and edges respectively) that distinguishes each bit from every other bit as a self-contained system. Hence, a standard basis can then act as both the **(1)** position and **(2)** orientation and **(3)** identifier of a specific bit, so both with respect to itself and the origin (mechanism) and with respect to each of the other bits of the self-contained system of 26 individual bits.

So, remarkably, the bit tracking equations can also track spin orientation of the particular bit when the position vector is additionally labelled by the respective colours (visible wavelengths). That is, the bit tracking equations with the basis-hat exponentials can then operate on the standard basis with the qualitative colour labels applied to the standard basis, which then also keeps track mathematically of the orientation as well as the position relative to the central mechanism (origin). This remarkable feature of maintaining the orientation in the vector is because the colours also point in the directions of the positions that they are in similarly as on a Rubiks cube, whilst the colour triplet maintains a bit's identity.

I have decided to call the **colour labelled position vectors** (of the standard basis) **colour vectors**. Alternatively, since every pure colour is associated with a unique wavelength, such a vector is alternatively called a **wavelength vector** [a **wavelength singlet** for a face, **wavelength doublet** for an edge, or **wavelength triplet** for a corner]. The colour labels can simply be considered scalar, or equivalently expressed in terms of wavelength. This means that the mathematics of the **colour vectors** is entirely mathematics of the standard basis and further generalizations are possible (but sufficient for the purposes here for Rubiks cubes). Further, the primary mechanism of this mathematics of basis variables is the cross, $\times$ [single basis component cross product] applied to the standard basis variables, that is, k -cross, j -cross, and i -cross which is what we formally thought of hencepreviously as complex numbers [and quartic roots of unity]. Complex numbers are the basis cross product triangle and the mathematics of this in all its glory is Rubik's marvellous invention, that is, a physical model of a closed system of interacting complex numbers (as complex exponentials). The colours of the 26 bits of a Rubiks cube, with the respective colour labels, are as follows:

The bits are	6 faces		12 edges		8 corners		
	1	r (red)	[+i-hat]	1	ry (red yellow)	1	rby (red blue yellow)
	2	o (orange)	[−i-hat]	2	oy (orange yellow)	2	oby (orange blue yellow)
	3	b (blue)	[+j-hat]	3	by (blue yellow)	3	rgy (red green yellow)
	4	g (green)	[−j-hat]	4	gy (green yellow)	4	ogy (orange green yellow)
	5	y (yellow)	[+k-hat]	5	rg (red blue)	5	rbw (red blue white)
	6	w (white)	[−k-hat]	6	ob (orange blue)	6	obw (orange blue white)
				7	rb (red green)	7	rgw (red green white)
				8	og (orange green)	8	ogw (orange green white)
				9	rw (red white)		
				10	ow (orange white)		
				11	bw (blue white)		
				12	gw (green white)		

These colour labels are then applied to the basis **position vector**, which are then **colour vectors**, and are consequently basis triplets that remarkably then keep track of which directions the bit is pointing (in addition to its position).

I have had to determine what colours to assign to what positions. This decision determines each bit's **colour vector** and the handedness (chirality) of the system. The colour scheme of a cube is standardised but the decision here is whether the cool colours (**b** and **g**) or the warm colours (**r** and **o**) are i -hat or j -hat [opposite for stars – hot are blue, cooler are orange and red]. The colours, and hence labels, **y** and **w** are the k -hat (z) axis faces: **y** the face of the positive pole (pointing up in z direction, $\hat{k} \mathbf{y}$) and **w** the face of the negative pole (pointing down $-\hat{k} \mathbf{w}$) where **y** without the hat is the yellow label and **w** is the white label. The **colour vectors** in this state are the home (solved) positions.

Here is what I have decided on;

y is on all the positive k -hats (top U) and **w** is on all the negative k -hats (bottom D)
r on positive i -hat (right R) **o** on negative i -hat (left L)
b on positive j -hat (front F) **g** on negative j -hat (back B)

... this is a left handed system which is that of electrons [although this is not why I choose a left handed system].

I have chosen a left handed system so that the i -hat direction is the x -axis which is what would be the y -axis in a right handed system. I chose this because R moves are the most common and I wanted this to correspond with i . F moves are then j . For a right handed system, these two are swapped in which case the most common would be j [which would be satisfactory as well I guess given that this is the complex number in electromagnetic theory]. I will be keeping with i for R moves for the time being, although to change to a right handed system, simply swap the labels of i -hat with those of j -hat.

In basis format, enumeration of each of the 26 unique colour labelled bits as (unprimed column) **colour vectors** is

6 **faces**: right $\hat{i} \mathbf{r}$, left $-\hat{i} \mathbf{o}$, front $\hat{j} \mathbf{b}$, back $-\hat{j} \mathbf{g}$, top $\hat{k} \mathbf{y}$, bottom $-\hat{k} \mathbf{w}$

12 **edges**:
 top layer $\hat{i} \mathbf{r} + \hat{k} \mathbf{y}, -\hat{i} \mathbf{o} + \hat{k} \mathbf{y}, \hat{j} \mathbf{b} + \hat{k} \mathbf{y}, -\hat{j} \mathbf{g} + \hat{k} \mathbf{y}$
 middle layer $\hat{i} \mathbf{r} + \hat{j} \mathbf{b}, -\hat{i} \mathbf{o} + \hat{j} \mathbf{b}, \hat{i} \mathbf{r} - \hat{j} \mathbf{g}, -\hat{i} \mathbf{o} - \hat{j} \mathbf{g}$
 bottom layer $\hat{i} \mathbf{r} - \hat{k} \mathbf{w}, -\hat{i} \mathbf{o} - \hat{k} \mathbf{w}, \hat{j} \mathbf{b} - \hat{k} \mathbf{w}, -\hat{j} \mathbf{g} - \hat{k} \mathbf{w}$

8 **corners**:
 top layer $\hat{i} \mathbf{r} + \hat{j} \mathbf{b} + \hat{k} \mathbf{y}, -\hat{i} \mathbf{o} + \hat{j} \mathbf{b} + \hat{k} \mathbf{y}, \hat{i} \mathbf{r} - \hat{j} \mathbf{g} + \hat{k} \mathbf{y}, -\hat{i} \mathbf{o} - \hat{j} \mathbf{g} + \hat{k} \mathbf{y}$
 bottom layer $\hat{i} \mathbf{r} + \hat{j} \mathbf{b} - \hat{k} \mathbf{w}, -\hat{i} \mathbf{o} + \hat{j} \mathbf{b} - \hat{k} \mathbf{w}, \hat{i} \mathbf{r} - \hat{j} \mathbf{g} - \hat{k} \mathbf{w}, -\hat{i} \mathbf{o} - \hat{j} \mathbf{g} - \hat{k} \mathbf{w}$

where, for example, $\hat{i} \mathbf{r} + \hat{j} \mathbf{b} + \hat{k} \mathbf{y}$ is the red (**r**), blue (**b**), and yellow (**y**) **corner bit colour vector** with **home (solved) position** in the top layer, i.e., **home positioned** at $\hat{i} + \hat{j} + \hat{k}$ (that is, $x=+1, y=+1$, and $z=+1$) ignoring the root 3 length from the origin (which is implicit in the scalar labels, similarly as a normalized set of electromagnetic ac quantities quoted in **RootMeanSquare**). In row position vector (primed) format

6 **faces**: right $(\mathbf{r}, 0, 0)$, left $(-\mathbf{o}, 0, 0)$, front $(0, \mathbf{b}, 0)$, back $(0, -\mathbf{g}, 0)$, top $(0, 0, \mathbf{y})$, bottom $(0, 0, -\mathbf{w})$

12 **edges**:
 top layer $(\mathbf{r}, 0, \mathbf{y}), (-\mathbf{o}, 0, \mathbf{y}), (0, \mathbf{b}, \mathbf{y}), (0, -\mathbf{g}, \mathbf{y})$
 middle layer $(\mathbf{r}, \mathbf{b}, 0), (-\mathbf{o}, \mathbf{b}, 0), (\mathbf{r}, -\mathbf{g}, 0), (-\mathbf{o}, -\mathbf{g}, 0)$
 bottom layer $(\mathbf{r}, 0, -\mathbf{w}), (-\mathbf{o}, 0, -\mathbf{w}), (0, \mathbf{b}, -\mathbf{w}), (0, -\mathbf{g}, -\mathbf{w})$

8 **corners**:
 top layer $(\mathbf{r}, \mathbf{b}, \mathbf{y}), (-\mathbf{o}, \mathbf{b}, \mathbf{y}), (\mathbf{r}, -\mathbf{g}, \mathbf{y}), (-\mathbf{o}, -\mathbf{g}, \mathbf{y})$
 bottom layer $(\mathbf{r}, \mathbf{b}, -\mathbf{w}), (-\mathbf{o}, \mathbf{b}, -\mathbf{w}), (\mathbf{r}, -\mathbf{g}, -\mathbf{w}), (-\mathbf{o}, -\mathbf{g}, -\mathbf{w})$

For the tracking of orientation by assigning colours [wavelengths] and hence **colour vectors** to the components as here, I think any form of the operators ($\pi/2$ basis-hat exponentials rotation matrices or the full rank diads such as $\hat{k} \hat{k}' + \hat{k} \times$ or $-\hat{i} \hat{j}' + \hat{j} \hat{i}' + \hat{k} \hat{k}'$) work similarly well (since they are alternative ways of expressing the same thing) since the labels are not part of the operators and the operators themselves are purely the mathematics of the matrix bases without any labelling (and only contain zeroes, ones, and negative ones) and is therefore a matter of personal preference for any particular context. A sequence of rotations are self-contained operators in-and-of-themselves before an application to (operation on) a **position vector** or **colour vector**, and consequently work equally well on (as well as independently from) either a **position vector** or a more informative **colour vector**.

Looking from above on the positive pole of an axis, that is, onto the page perpendicular to the axis, the rotations are **counter-clockwise (ccw $\mathcal{O}$)** as per standard mathematics of the complex plane (equivalently, R^2 vector algebra with $i = \hat{k} \times$). This, as a point of relevance, is opposite to cube algorithms for R, U, and F (Dan Harris, Super Strategies for Speedcubers, Stirling Publishing; the book from which I learned cubing and speedcubing) which are **clockwise (cw $\mathcal{U}$)** operators which are therefore conjugate rotational directions to mathematical convention. The R, U, and F rotations are the rotations that I am focussing on although note that the L(left), D(down), and B(back) are non-conjugate **ccw $\mathcal{O}$** rotations since the Rubiks algorithms swap spin direction on the opposite poles (similar to the Coriolis force in the Northern and Southern hemispheres). This needs to be borne in mind when replicating with

maths and with the physical cube and making them correspond as here. This can be quite frustrating initially despite being simple in concept (that left and right swap upon being inverted for example). The one thing I can say almost absolutely, the maths works.

Examples: the three examples next return the bit to the original position and also leave the orientation unchanged, that is, they are identity for these bits [but not the other 25 bits most of which (any on the same slice) have changed positions][which is a very interesting aspect of Rubiks cubes in that it is not possible to change the position of one bit without changing the position of other bits (of the slice around the rotation axis) which is a mathematical feature which is not model-able with purely scalar mathematics]. It is worthwhile actually doing the rotations on a cube as I have with these algorithms (as another way to check the correctness of the mathematical algorithms, and visa-versa):

(1) center bit example (red face bit which is situated on the right side in home [solved] position): $k i j'$

$$e^{\frac{\pi}{2} \hat{k} \times} e^{\frac{\pi}{2} \hat{i} \times} \left(e^{\frac{\pi}{2} \hat{j} \times} \right)^T \begin{pmatrix} \mathbf{r} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}^T \begin{pmatrix} \mathbf{r} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbf{r} \\ 0 \\ 0 \end{pmatrix} \quad (\text{these are middle moves})$$

(2) edge bit example: (red blue edge bit which is situated on the front right middle layer in home [solved] position): $i k^2 i' j^2$

$$e^{\frac{\pi}{2} \hat{i} \times} \left(e^{\frac{\pi}{2} \hat{k} \times} \right)^2 \left(e^{\frac{\pi}{2} \hat{i} \times} \right)^T \left(e^{\frac{\pi}{2} \hat{j} \times} \right)^2 \begin{pmatrix} \mathbf{r} \\ \mathbf{b} \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}^T \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}^2 \begin{pmatrix} \mathbf{r} \\ \mathbf{b} \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbf{r} \\ \mathbf{b} \\ 0 \end{pmatrix}$$

(3) corner bit example (red, blue, and yellow corner bit which is situated on the front right top layer in home position): $k j k' i$

$$e^{\frac{\pi}{2} \hat{k} \times} e^{\frac{\pi}{2} \hat{j} \times} \left(e^{\frac{\pi}{2} \hat{k} \times} \right)^T e^{\frac{\pi}{2} \hat{i} \times} \begin{pmatrix} \mathbf{r} \\ \mathbf{b} \\ \mathbf{y} \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}^T \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{r} \\ \mathbf{b} \\ \mathbf{y} \end{pmatrix} = \begin{pmatrix} \mathbf{r} \\ \mathbf{b} \\ \mathbf{y} \end{pmatrix}$$

These examples happen to also leave the bit orientations [**colour vectors**] unchanged [since they are identity moves] similarly as the earlier examples for **position vectors**. The results can then be obtained for **position vectors** by throwing away the colour information, by setting $\mathbf{r}$, $\mathbf{b}$, and $\mathbf{y}$ to unity for example (in which case the **colour vectors** revert to **position vectors**). Alternatively, faces set to 1, edges set to $1/\sqrt{2}$, and corners set to $1/\sqrt{3}$ for a normalized set of position vectors. I initially chose these examples (sets of moves) to keep things simple. I hence needed some other examples to further demonstrate orientation. This is obviously not difficult, other than with indecision, given that there is practically an infinite number of combinations. To keep things simple, I have focused on $i j k$ for the examples of the three types of bits and the associated **colour vectors** to track both position and bit orientation, or more precisely, $\exp(i \pi/2 \times)$, $\exp(j \pi/2 \times)$, and $\exp(k \pi/2 \times)$, to include the rotational eigenvectors. That is, these examples will apply $\exp(i \pi/2 \times)$ then $\exp(j \pi/2 \times)$ and then $\exp(k \pi/2 \times)$ to the **colour vectors** of the three types of bit (a **face**, an **edge**, and a **corner**).

(1) center bit example (red face bit which is situated on the right side in home [solved] position): $i j k$

the position vector for the home [solved] position of this bit is $\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$ and hence the **colour vector** is $\begin{pmatrix} \mathbf{r} & 0 & 0 \end{pmatrix}$

$$e^{\frac{\pi}{2}\hat{i}\times} e^{\frac{\pi}{2}\hat{j}\times} e^{\frac{\pi}{2}\hat{k}\times} \begin{pmatrix} \mathbf{r} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{r} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \mathbf{r} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ +1\mathbf{r} \end{pmatrix} \quad \text{from } \begin{pmatrix} +1\mathbf{r} \\ 0 \\ 0 \end{pmatrix}$$

what was this on a Rubik's cube? The first rotation, ccw $\mathfrak{U}$ around i -hat, left the red face's position unchanged [a rotation of a pole of the x -axis] and hence its **colour vector** unchanged. The second rotation, ccw $\mathfrak{U}$ around j -hat, moves the red face, which is an axis, to the top layer, that is, the red face becomes the positive pole of the axis pointing in the k -hat direction [the z -axis]. The third and last rotation, ccw $\mathfrak{U}$ around k -hat, left this position unchanged since this is the direction in which the face is pointing [hence another rotation of a pole of an axis, now the z -axis]. The third rotation hence leaves the red face pointing in the positive k -hat direction, that is, a positive 1 coefficient of the red label in the k -hat position of the **colour vector**, i.e., a **colour vector** in the state $(0 \ 0 \ +1\mathbf{r})'$. In

other words the $i j k$ operator applied to the operand $\mathbf{r}\hat{i} = (+1\mathbf{r} \ 0 \ 0)'$ converted the **colour vector** to $\mathbf{r}\hat{k} = (0 \ 0 \ +1\mathbf{r})'$

$$(\hat{i}\hat{i}'+\hat{i}\times)(\hat{j}\hat{j}'+\hat{j}\times)(\hat{k}\hat{k}'+\hat{k}\times)\mathbf{r}\hat{i} = \mathbf{r}\hat{k}$$

(2) edge bit example: (red blue edge bit which is situated on the front right middle layer in home [solved] position): $i j k$

the position vector for the home [solved] position of this bit is $(1 \ 1 \ 0)$ and hence the **colour vector** is $(\mathbf{r} \ \mathbf{b} \ 0)$

$$e^{\frac{\pi}{2}\hat{i}\times} e^{\frac{\pi}{2}\hat{j}\times} e^{\frac{\pi}{2}\hat{k}\times} \begin{pmatrix} \mathbf{r} \\ \mathbf{b} \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{r} \\ \mathbf{b} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -\mathbf{b} \\ \mathbf{r} \end{pmatrix} = \begin{pmatrix} 0 \\ -1\mathbf{b} \\ +1\mathbf{r} \end{pmatrix} \quad \text{from } \begin{pmatrix} +1\mathbf{r} \\ +1\mathbf{b} \\ 0 \end{pmatrix}$$

what was this on a Rubik's cube? The first rotation, **ccw** $\mathfrak{U}$ around i -hat, rotates the edge from the front right middle to the bottom right middle as an outer layer rotation with the red face still pointing in the i -hat direction and the blue face now pointing down in the negative k -hat direction. The second rotation, **ccw** $\mathfrak{U}$ around j -hat, moves the bit as a middle layer rotation back to the top starting position but with the red and blue orientation swapped. The third and last rotation, **ccw** $\mathfrak{U}$ around k -hat, moves the edge from the top right to the top back. This leaves the **colour vector** of the edge piece with the red face pointing up in the positive k -hat direction and the blue face pointing backwards in the negative j -hat direction, that is, in the state $(0 \ -1\mathbf{b} \ +1\mathbf{r})'$. In other words the $i j k$ operator

applied to the operand $\mathbf{r}\hat{i} + \mathbf{b}\hat{j} = (+1\mathbf{r} \ +1\mathbf{b} \ 0)'$ converted the **colour vector** to $-\mathbf{b}\hat{j} + \mathbf{r}\hat{k} = (0 \ -1\mathbf{b} \ +1\mathbf{r})'$

$$(\hat{i}\hat{i}'+\hat{i}\times)(\hat{j}\hat{j}'+\hat{j}\times)(\hat{k}\hat{k}'+\hat{k}\times)(\mathbf{r}\hat{i} + \mathbf{b}\hat{j}) = -\mathbf{b}\hat{j} + \mathbf{r}\hat{k}$$

(3) corner bit example (red, blue, and yellow corner bit which is situated on the front right top layer in home [solved] position): $i j k$

the position vector for the home [solved] position of this bit is $(1 \ 1 \ 1)$ and hence the **colour vector** is $(\mathbf{r} \ \mathbf{b} \ \mathbf{y})$

$$e^{\frac{\pi}{2}\hat{i}\times} e^{\frac{\pi}{2}\hat{j}\times} e^{\frac{\pi}{2}\hat{k}\times} \begin{pmatrix} \mathbf{r} \\ \mathbf{b} \\ \mathbf{y} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{r} \\ \mathbf{b} \\ \mathbf{y} \end{pmatrix} = \begin{pmatrix} \mathbf{y} \\ -\mathbf{b} \\ \mathbf{r} \end{pmatrix} = \begin{pmatrix} +1\mathbf{y} \\ -1\mathbf{b} \\ +1\mathbf{r} \end{pmatrix} \quad \text{from } \begin{pmatrix} +1\mathbf{r} \\ +1\mathbf{b} \\ +1\mathbf{y} \end{pmatrix}$$

what was this on a Rubik's cube? The first rotation, **ccw** $\mathfrak{U}$ around i -hat, rotates the corner from the top front right to the bottom front right keeping the red face with the same orientation in the i -hat direction whilst changing the yellow face to the positive j -hat direction and the blue face pointing down in the negative k -hat direction. The second rotation, **ccw** $\mathfrak{U}$ around j -hat, moves the corner back up to the starting position but with the orientation changed, red pointing up, yellow pointing out front and blue pointing right. The third and last rotation, **ccw** $\mathfrak{U}$ around k -hat, moves the corner from the top front right to the top back right with red still pointing up, yellow pointing right, and blue pointing down. This leaves the **colour vector** in the state $(+1\mathbf{y} \ -1\mathbf{b} \ +1\mathbf{r})'$. In other words the $i j k$ operator applied to the operand $\mathbf{r}\hat{i} + \mathbf{b}\hat{j} + \mathbf{y}\hat{k} = (+1\mathbf{r} \ +1\mathbf{b} \ +1\mathbf{y})'$ converted the **colour vector** to $\mathbf{y}\hat{i} - \mathbf{b}\hat{j} + \mathbf{r}\hat{k} = (+1\mathbf{y} \ -1\mathbf{b} \ +1\mathbf{r})'$

$$(\hat{i}\hat{i}' + \hat{i} \times) (\hat{j}\hat{j}' + \hat{j} \times) (\hat{k}\hat{k}' + \hat{k} \times) (\mathbf{r}\hat{i} + \mathbf{b}\hat{j} + \mathbf{y}\hat{k}) = \mathbf{y}\hat{i} - \mathbf{b}\hat{j} + \mathbf{r}\hat{k}$$

Note that the overall $i j k$ operator in matrix format, which could obviously be applied to the **colour vector** (or position vector), is

$$(\hat{i}\hat{i}' + \hat{i} \times) (\hat{j}\hat{j}' + \hat{j} \times) (\hat{k}\hat{k}' + \hat{k} \times) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

which looks like a j -hat $\pi/2$ exponential in that it is an anti-diagonal matrix, but has a different position for the inverting negative. This is an operator in-and-of-itself. It is actually the quaternion-like operator for j -hat. This and the j -hat exponential and the conjugate form the set

$$e^{\frac{\pi}{2}\hat{j} \times} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad e^{\frac{\pi}{2}\hat{i} \times} e^{\frac{\pi}{2}\hat{j} \times} e^{\frac{\pi}{2}\hat{k} \times} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad e^{-\frac{\pi}{2}\hat{j} \times} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

It seems simple, yet difficult, similarly as a Rubiks cube to a beginner. Once it is all sorted though, it is simple yet sophisticated whilst also intuitive.

As a last point for this section, I have noted that a basis is equivalent to memory addresses in c-programming, and the pointer functionality of c and c++ is what makes these programming languages so powerful, notably, the ability to manipulate memory addresses (the computer memory basis) directly.

It is also the most difficult aspect to learn of c-programming, so a **basis interpretation** (of pointers) might help with this, showing why and how it is such a powerful facet of these languages (c and c++).

I have written a simple c program, on the right, that just attempts to highlight this, notably, that pointers are equivalent to basis variables.

```
C testfun.c > ...
1  #include <stdio.h>
2  #include <stdlib.h>
3
4  int main()
5  {
6      int x, y, z; //declare three coordinate integers
7
8      printf("Enter three coordinate integers: \n");
9      scanf("%d", &x); //scan integers into memory
10     scanf("%d", &y); //i.e., assign to pre-existing basis
11     scanf("%d", &z);
12
13     int* ihat = malloc(sizeof(int)); //establish new basis
14     int* jhat = malloc(sizeof(int)); //to hold
15     int* khat = malloc(sizeof(int)); //scalar integers
16
17     ihat = &x; //assign pre-existing memory
18     jhat = &y; //to new basis variables
19     khat = &z; //i.e., to the memory addresses
20
21     //dereference basis (pointer) with * for scalar data
22     printf("\nThe first number (x-data) is %d\n", *ihat);
23     printf("Stored at address (i-hat) %p\n\n", ihat);
24
25     //hence memory address (&) is basis placeholder
26     printf("The second number (y-data) is %d\n", *jhat);
27     printf("Stored at address (j-hat) %p\n\n", jhat);
28
29     printf("The third number (z-data) is %d\n", *khat);
30     printf("Stored at address (k-hat) %p\n\n", khat);
31
32     return(0);
33 }
```

The output shows that it doesn't do anything fancy, rather just makes clear a basis interpretation of c-pointers and computer memory. I will be extending all of the R^3 theory in this document, with $i = \hat{k} \times$, to R^n the next time I have some free time [quite a bit of time because it will probably be about the size of (yet) another PhD].

I have now sorted out how to do this, that is, how to make the theory $i = \hat{k} \times$ applicable to R^n . All of the theory of linear algebra and complex numbers can now be applied directly to computer memory. We will essentially be able to turn computer memory into very large Rubiks cubes which will potentially greatly increase the sophistication of the now very large processors (with the computer clock providing the time-base for dynamics).

which outputs

```
PROBLEMS  TERMINAL  ...  powershell + v
PS C:\Users\aaaron\MyC> gcc testfun.c
PS C:\Users\aaaron\MyC> ./a
Enter three coordinate integers:
3 5 7

The first number (x-data) is 3
Stored at address (i-hat) 000000DA3A9FF6F4

The second number (y-data) is 5
Stored at address (j-hat) 000000DA3A9FF6F0

The third number (z-data) is 7
Stored at address (k-hat) 000000DA3A9FF6EC
```

Vector Cross Product (as distinct from single component cross product of a cross product triangle)

If a complex number $i = \sqrt{-1}$ is the cross ($\times$) of a single basis vector k -hat, and a cross product consists of the cross ($\times$) of three such basis vectors i -hat, j -hat, and k -hat, what is a cross product of two vectors under the circumstances that k -cross was still considered complex? The cross product would then be considered a quaternion. The cross product of two vectors is complex numbers on the three mutually perpendicular axes. It is known definitively to be entirely real and no complex field is required for this. It is quaternion-like in that the operating vector is a linear combination of the 3 complex quantities i , j , and k , but these are known not be complex numbers but the basis vector rotation operators, k -cross, j -cross, and i -cross (formally thought of as scalar $i = \sqrt{-1}$, $j = \sqrt{-1}$, and $k = \sqrt{-1}$, although not scalar, but operator with these scalar properties once the standard basis is appended with $\times$). The weighting of the linear combination is then by the three scalar components of the operating vector:

$$\vec{x} \times \vec{y} = (x_1 \hat{i} + x_2 \hat{j} + x_3 \hat{k}) \times (y_1 \hat{i} + y_2 \hat{j} + y_3 \hat{k}) = (x_1 \hat{i} \times + x_2 \hat{j} \times + x_3 \hat{k} \times)(y_1 \hat{i} + y_2 \hat{j} + y_3 \hat{k})$$

where $(x_1 \hat{i} + x_2 \hat{j} + x_3 \hat{k}) \times (\cdot) = x_1 \hat{i} \times (\cdot) + x_2 \hat{j} \times (\cdot) + x_3 \hat{k} \times (\cdot)$ is then the (quaternion-like) operator that operates on the vector $y_1 \hat{i} + y_2 \hat{j} + y_3 \hat{k}$, that is, operates on the basis vectors that generates the span for, and consequently placemarks (assigns a memory address), the scalar components y_1 , y_2 , and y_3 . It demonstrates that $\times$ is distributive independently of i -hat, j -hat, or k -hat, as a binary operator, for i -hat, j -hat, or k -hat. It is also known that vectors in the cross product anti-commute, so the cross product of the x -vector with the y -vector is the negative of the cross product of the y -vector with the x -vector. This gives extra details on the nature of $\times$ and the nature of $\times$ when it is coupled with i -hat, j -hat, and k -hat. If $i = \sqrt{-1}$ is defined as the operator k -cross, then $i = \hat{k} \times = (\hat{k} \times) = (\hat{k})(\times)$ and then $d(\hat{k} \times) / d\hat{k} = \times$. The binary operator $\times$ introduces a discrete quarter rotational spin and, further, the basis variable to which it multiplies is the axis around which the rotational spin occurs. Continuing with the cross product:

$$\begin{aligned} \vec{x} \times \vec{y} &= (x_1 \hat{i} \times + x_2 \hat{j} \times + x_3 \hat{k} \times)(y_1 \hat{i} + y_2 \hat{j} + y_3 \hat{k}) \\ &= x_1 \hat{i} \times (y_1 \hat{i} + y_2 \hat{j} + y_3 \hat{k}) + x_2 \hat{j} \times (y_1 \hat{i} + y_2 \hat{j} + y_3 \hat{k}) + x_3 \hat{k} \times (y_1 \hat{i} + y_2 \hat{j} + y_3 \hat{k}) \end{aligned}$$

which demonstrates that $\times$ is distributive once coupled with a basis vector to form the ‘complex-like number’ and hence form the ‘quaternion-like number’ when each of k -cross, j -cross, and i -cross is considered as a weighted sum (of the scalars x_1 , x_2 , and x_3) into a single vector operator.

$$\begin{aligned} \vec{x} \times \vec{y} &= x_1 \hat{i} \times (y_1 \hat{i} + y_2 \hat{j} + y_3 \hat{k}) + x_2 \hat{j} \times (y_1 \hat{i} + y_2 \hat{j} + y_3 \hat{k}) + x_3 \hat{k} \times (y_1 \hat{i} + y_2 \hat{j} + y_3 \hat{k}) \\ &= \left(\cancel{x_1 y_1 (\hat{i} \times \hat{i})} + x_1 y_2 (\hat{i} \times \hat{j}) + x_1 y_3 (\hat{i} \times \hat{k}) \right) + \left(x_2 y_1 (\hat{j} \times \hat{i}) + \cancel{x_2 y_2 (\hat{j} \times \hat{j})} + x_2 y_3 (\hat{j} \times \hat{k}) \right) + \left(x_3 y_1 (\hat{k} \times \hat{i}) + x_3 y_2 (\hat{k} \times \hat{j}) + \cancel{x_3 y_3 (\hat{k} \times \hat{k})} \right) \\ &= \left(\cancel{x_1 y_1 (\hat{i} \times \hat{i})} + x_1 y_2 \hat{k} - x_1 y_3 \hat{j} \right) + \left(-x_2 y_1 \hat{k} + \cancel{x_2 y_2 (\hat{j} \times \hat{j})} + x_2 y_3 \hat{i} \right) + \left(x_3 y_1 \hat{j} - x_3 y_2 \hat{i} + \cancel{x_3 y_3 (\hat{k} \times \hat{k})} \right) \\ &= \hat{i}(x_2 y_3 - x_3 y_2) + \hat{j}(x_3 y_1 - x_1 y_3) + \hat{k}(x_1 y_2 - x_2 y_1) \end{aligned}$$

The workings of the stock standard (plain vanilla) cross product thus shows and demonstrates definitively that there is a different complex number [real basis curl and an operator, **not a number**] for each axis, and shows how each complex number [real basis curl] works with itself and with the others (j -cross and i -cross for k -cross for example). It also shows that there is no such things as a quaternion field, similarly as there is no such things as a complex field. A complex number is a single component cross product,

whilst a quaternion is an arbitrary position vector appended with $\times$ which hence does not conform to a cross product triangle, i.e., an arbitrary vector cross product.

Hence, this is how a proper quaternion-like operator would work, if there was one. There is not since it is no longer complex, so is not a quaternion as to how it has been defined, similarly as a complex number: there is no extended field to complex numbers or further extension to quaternions, this is all rotational cross products and all calculus in the reals R^3 . It operates **exactly** as it does as the cross product. Defining the complex numbers as basis curls obviates the requirement for complex numbers and quaternion theory as some extended esoteric field [since they aren't]. These notions can be adapted into how we modify the maths in conversion over to real vector calculus, as the terms [complex number and quaternion, but especially quaternion] allow for an extra degree of freedom in how the mathematics is described. The term 'quaternion' is a good one. As I have mentioned, I like to think of **Rubiks cubes** and the many **derivatives of Rubiks cubes** (pyraminx, megaminx, skewb, and so on) as **Platonic quaternions** which stresses that these toys, and what they describe in their motion and configuration, are mathematical. The mechanism in Platonic quaternions (Rubiks cubes and their derivatives) are the same as a set of coordinate axes and hence the vector basis of the coordinate system defined by those axes. It is also a good word for 'quarter rotations' which is what basis curls are [note that it is the k -hat exponential and its conjugate in the context of Euler's formula that converts the basis-rotations to continuous infinite roots of unity as a function of continuous angle θ].

Theorem: A quaternion is equivalent to a standard vector cross-product, similarly as a complex number is equivalent to the single component cross product $i = \hat{k} \times$

This is an arbitrary R^3 basis vector comprised of a linear combination of i -hat, j -hat, and k -hat, appended with cross $\times$.

The important point being that a standard arbitrary vector product does not follow a cross product triangle, the individual components of the standard vector product do, those of $i = \hat{k} \times$, $\hat{j} \times$, and $\hat{i} \times$.

Corollary: There is no quaternion field. The quaternion field is equivalent to the R^3 vector space.

Diadic expansion (of the vector basis into a matrix basis) then expresses the cross product as either

$$\vec{x} \times \vec{y} = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \quad \text{or} \quad \vec{x} \times \vec{y} = -\vec{y} \times \vec{x} = \begin{pmatrix} 0 & y_3 & -y_2 \\ -y_3 & 0 & y_1 \\ y_2 & -y_1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

noting that the diadic expanded matrix is not positive definite. The minimum polynomial basis factorization of this matrix is the leading vector appended with cross $\times$. The cross product implies that $\times$ is distributed with i -hat, j -hat, and k -hat as a binary operator for the basis vectors and hence has an independent existence as an operator. Defined in this way, the cross product of a vector with another vector (both of which are generated by the basis vectors) is the distributive product of the three complex numbers (real basis curls) onto the basis variables of the other, that is, the cross $\times$ converts the first vector into a basis curl (quaternion-like) operator (of three complex numbers [basis curls] i -cross, j -cross, and k -cross) which then operates on the second vector comprised of the weighted sum of i -hat, j -hat, and k -hat. Viewed in this way, the mathematical emphasis is placed on the basis, rather than the scalars which are labelled and placemarked in space (R^3) by the basis.

Thus re-defining the quaternion as an operator [an arbitrary three component R^3 vector appended with $\times$] similarly as redefining the complex number as the operator $i = \hat{k} \times$ rather than the algebra quantity $i = \sqrt{-1}$. A quaternion is the cross product, as an operator. A quaternion is ordinarily defined [from a standard definition from, say, Penguin Reference Dictionary of Mathematics, Daintith and Nelson] as

$$x_0 + x_1 i + x_2 j + x_3 k \quad \text{where } i = \sqrt{-1}, j = \sqrt{-1}, \text{ and } k = \sqrt{-1}$$

With continuity of argument as for $i = \hat{k} \times$, this is actually the vector cross product and is an operator similarly as $i = \hat{k} \times$. Redefining the components as in the first part of this document $i = \hat{k} \times$, $j = \hat{j} \times$, and $k = \hat{i} \times$ for spin rotations around the z, y, and x axes (generated by k -hat, j -hat, and i -hat) respectively, on the x-y plane, x-z, and y-z planes. This then demonstrates that it is a cross product, considered as an operator, and explains why there is a constant term. Now given that quaternion-like mapping is a real R^3 stock standard cross product operator in a vector volume R^3 [and not an extended field of C], the quaternion is then

$$\vec{x}(\cdot) = x_0 + x_1 \hat{k} \times (\cdot) + x_2 \hat{j} \times (\cdot) + x_3 \hat{i} \times (\cdot) = (x_0 + x_1 \hat{k} \times + x_2 \hat{j} \times + x_3 \hat{i} \times)(\cdot)$$

In the i -hat direction

$$\vec{x} \hat{i} = (x_0 + x_1 \hat{k} \times + x_2 \hat{j} \times + x_3 \hat{i} \times) \hat{i} = x_0 \hat{i} + x_1 \hat{k} \times \hat{i} + x_2 \hat{j} \times \hat{i} + x_3 \cancel{\hat{i} \times \hat{i}} = x_0 \hat{i} + x_1 \hat{j} - x_2 \hat{k}$$

which is then the vector $\vec{x} \hat{i} = (x_0, x_1, -x_2)$. In the j -hat direction

$$\vec{x} \hat{j} = (x_0 + x_1 \hat{k} \times + x_2 \hat{j} \times + x_3 \hat{i} \times) \hat{j} = x_0 \hat{j} + x_1 \hat{k} \times \hat{j} + x_2 \cancel{\hat{j} \times \hat{j}} + x_3 \hat{i} \times \hat{j} = -x_1 \hat{i} + x_0 \hat{j} + x_3 \hat{k}$$

which is the vector $\vec{x} \hat{j} = (-x_1, x_0, x_3)$. In the k -hat direction

$$\vec{x} \hat{k} = (x_0 + x_1 \hat{k} \times + x_2 \hat{j} \times + x_3 \hat{i} \times) \hat{k} = x_0 \hat{k} + x_1 \cancel{\hat{k} \times \hat{k}} + x_2 \hat{j} \times \hat{k} + x_3 \hat{i} \times \hat{k} = x_2 \hat{i} - x_3 \hat{j} + x_0 \hat{k}$$

which is the vector $\vec{x} \hat{k} = (x_2, -x_3, x_0)$. If these vectors are then weighted (scaled) by the components (scalars) of another vector y_1, y_2 , and y_3 and summed, which essentially weight the i -hat, j -hat, and k -hat that the quaternion has operated on and distributed over, then

$$\vec{x} y_1 \hat{i} = y_1 (x_0 \hat{i} + x_1 \hat{j} - x_2 \hat{k}) = x_0 y_1 \hat{i} + x_1 y_1 \hat{j} - x_2 y_1 \hat{k}$$

$$\vec{x} y_2 \hat{j} = y_2 (-x_1 \hat{i} + x_0 \hat{j} + x_3 \hat{k}) = -x_1 y_2 \hat{i} + x_0 y_2 \hat{j} + x_3 y_2 \hat{k}$$

$$\vec{x} y_3 \hat{k} = y_3 (x_2 \hat{i} - x_3 \hat{j} + x_0 \hat{k}) = x_2 y_3 \hat{i} - x_3 y_3 \hat{j} + x_0 y_3 \hat{k}$$

so then

$$\begin{aligned} (x_0 + x_1 \hat{k} \times + x_2 \hat{j} \times + x_3 \hat{i} \times) (y_1 \hat{i} + y_2 \hat{j} + y_3 \hat{k}) &= (x_0 y_1 \hat{i} + x_1 y_1 \hat{j} - x_2 y_1 \hat{k}) + (-x_1 y_2 \hat{i} + x_0 y_2 \hat{j} + x_3 y_2 \hat{k}) + (x_2 y_3 \hat{i} - x_3 y_3 \hat{j} + x_0 y_3 \hat{k}) \\ &= \hat{i} (x_0 y_1 - x_1 y_2 + x_2 y_3) + \hat{j} (x_1 y_1 + x_0 y_2 - x_3 y_3) + \hat{k} (-x_2 y_1 + x_3 y_2 + x_0 y_3) \end{aligned}$$

which is a cross product but with the curl (non-diagonal components) permuted and three terms in each component rather than two, due to the constant in the operator. With a diadic expansion, this is

$$\begin{pmatrix} x_0 & -x_1 & x_2 \\ x_1 & x_0 & -x_3 \\ -x_2 & x_3 & x_0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

so that the first term in the cross product can be expressed as a matrix operator as

$$\begin{pmatrix} x_0 & -x_1 & x_2 \\ x_1 & x_0 & -x_3 \\ -x_2 & x_3 & x_0 \end{pmatrix} (\cdot) = x_0 I + \begin{pmatrix} 0 & -x_1 & x_2 \\ x_1 & 0 & -x_3 \\ -x_2 & x_3 & 0 \end{pmatrix} (\cdot) = \left(x_0 I + \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \times \right) (\cdot)$$

which is that of a cross product with **(1)** a permutation of the non-diagonal components and **(2)** the constant placed along the diagonals making the matrix positive definite; this is because appending the standard basis with $\times$ converts it into an R^3 basis operator, and the addition of the constant is then equivalent to adding a scalar multiple of the identity I (as demonstrated by dyadic expansion).

Note also (and conversely of the diadic expansion) that $x_0 + x_1 \hat{k} \times + x_2 \hat{j} \times + x_3 \hat{i} \times$ is the minimum polynomial basis factorisation for this matrix [a three component curl plus a diagonal] and is alternatively obtained diadically with the diadically expanded operators of $\hat{k} \times$, $\hat{j} \times$, and $\hat{i} \times$

$$x_0 + x_1 \hat{k} \times + x_2 \hat{j} \times + x_3 \hat{i} \times = x_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + x_1 \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} x_0 & -x_1 & x_2 \\ x_1 & x_0 & -x_3 \\ -x_2 & x_3 & x_0 \end{pmatrix}$$

For correspondence with the cross product (in notation and mapping of vector components), using a slightly different definition of the quaternion-like operator by permuting the spin axes, the order for which seems obvious but contrary to considering the standard plane as x and y with the rotation around the latent z axis. Redefining the components as in the first part of this document but with the permutation of spin axes for $i = \hat{i} \times$, $j = \hat{j} \times$, and $k = \hat{k} \times$ for spin rotations around the x , y , and z axes (generated by i -hat, j -hat, and k -hat) respectively, on the y - z plane, x - z , and x - y planes

$$\begin{aligned} \vec{x}(\cdot) &= x_0 + x_1 \hat{i} \times (\cdot) + x_2 \hat{j} \times (\cdot) + x_3 \hat{k} \times (\cdot) = (x_0 + x_1 \hat{i} \times + x_2 \hat{j} \times + x_3 \hat{k} \times) (\cdot) \\ \vec{x} \hat{i} &= (x_0 + x_1 \hat{i} \times + x_2 \hat{j} \times + x_3 \hat{k} \times) \hat{i} = x_0 \hat{i} + x_1 \cancel{\hat{i} \times \hat{i}} + x_2 \hat{j} \times \hat{i} + x_3 \hat{k} \times \hat{i} = x_0 \hat{i} + x_3 \hat{j} - x_2 \hat{k} \\ \vec{x} \hat{j} &= (x_0 + x_1 \hat{i} \times + x_2 \hat{j} \times + x_3 \hat{k} \times) \hat{j} = x_0 \hat{j} + x_1 \hat{i} \times \hat{j} + x_2 \cancel{\hat{j} \times \hat{j}} + x_3 \hat{k} \times \hat{j} = -x_3 \hat{i} + x_0 \hat{j} + x_1 \hat{k} \\ \vec{x} \hat{k} &= (x_0 + x_1 \hat{i} \times + x_2 \hat{j} \times + x_3 \hat{k} \times) \hat{k} = x_0 \hat{k} + x_1 \hat{i} \times \hat{k} + x_2 \hat{j} \times \hat{k} + x_3 \cancel{\hat{k} \times \hat{k}} = x_2 \hat{i} - x_1 \hat{j} + x_0 \hat{k} \end{aligned}$$

and weighting these vectors by the scalar components of another vector y_1 , y_2 , and y_3 and summing (again, which essentially weights the i -hat, j -hat, and k -hat that the cross product has operated on and distributed over), then

$$\begin{aligned}\bar{x} y_1 \hat{i} &= y_1 (x_0 + x_1 \hat{i} \times + x_2 \hat{j} \times + x_3 \hat{k} \times) \hat{i} = x_0 y_1 \hat{i} + x_3 y_1 \hat{j} - x_2 y_1 \hat{k} \\ \bar{x} y_2 \hat{j} &= y_2 (x_0 + x_1 \hat{i} \times + x_2 \hat{j} \times + x_3 \hat{k} \times) \hat{j} = -x_3 y_2 \hat{i} + x_0 y_2 \hat{j} + x_1 y_2 \hat{k} \\ \bar{x} y_3 \hat{k} &= y_3 (x_0 + x_1 \hat{i} \times + x_2 \hat{j} \times + x_3 \hat{k} \times) \hat{k} = x_2 y_3 \hat{i} - x_1 y_3 \hat{j} + x_0 y_3 \hat{k}\end{aligned}$$

so that

$$\begin{aligned}(x_0 + x_1 \hat{i} \times + x_2 \hat{j} \times + x_3 \hat{k} \times)(y_1 \hat{i} + y_2 \hat{j} + y_3 \hat{k}) &= (x_0 y_1 \hat{i} + x_3 y_1 \hat{j} - x_2 y_1 \hat{k}) + (-x_3 y_2 \hat{i} + x_0 y_2 \hat{j} + x_1 y_2 \hat{k}) + (x_2 y_3 \hat{i} - x_1 y_3 \hat{j} + x_0 y_3 \hat{k}) \\ &= \hat{i} (x_0 y_1 - x_3 y_2 + x_2 y_3) + \hat{j} (x_3 y_1 + x_0 y_2 - x_1 y_3) + \hat{k} (-x_2 y_1 + x_1 y_2 + x_0 y_3)\end{aligned}$$

which generalizes the cross product (with the diagonal making it full rank) and is entirely real since the terms **(1)** that don't contain x_0 are the cross product and **(2)** those that do add an additional degree of freedom to the resultant (down the diagonal). This means that it is possible to offset both a complex number and a quaternion [although are entirely real vector operators] similarly as, say a k -hat exponential (with **cosk**), and have an entirely real result in a real R^3 vector space. This scales the y vector by x_0 and then adds to this the cross product, i.e.,

$$\begin{aligned}(x_0 + x_1 \hat{i} \times + x_2 \hat{j} \times + x_3 \hat{k} \times)(y_1 \hat{i} + y_2 \hat{j} + y_3 \hat{k}) &= x_0 (y_1, y_2, y_3) + (x_1, x_2, x_3) \times (y_1, y_2, y_3) \\ &= (x_0 y_1, x_0 y_2, x_0 y_3) + (x_1, x_2, x_3) \times (y_1, y_2, y_3) \\ &= (x_0 y_1, x_0 y_2, x_0 y_3) + (x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1) \\ &= (x_0 y_1 - x_3 y_2 + x_2 y_3, x_3 y_1 + x_0 y_2 - x_1 y_3, -x_2 y_1 + x_1 y_2 + x_0 y_3) \\ &= \hat{i} (x_0 y_1 - x_3 y_2 + x_2 y_3) + \hat{j} (x_3 y_1 + x_0 y_2 - x_1 y_3) + \hat{k} (-x_2 y_1 + x_1 y_2 + x_0 y_3)\end{aligned}$$

note that the constant term is similar to the cosine (**cosk**) in the k -hat exponential. This constant cannot be placed in the y vector (the operand, unless it is converted to the operator) as it must be part of the operator since a constant cannot exist outside of i -hat, j -hat, and k -hat. I was having trouble with the Rubiks cube position tracking equations because when I would cross out a k -hat component for a k -hat spin (for lev or dex), I had to add it back again to get a non-zero position for that component (this is the eigenvalue). **This means that Rubiks cubes really are quaternions [cross product operators][although taken component-wise]. Platonic quaternions, because they are also Platonic solids.** Quaternions are entirely real in R^3 when defined in the context of real basis rotation operators, that is, a linear combination of the components of the standard basis appended with $\times$. Here is the new definition of a quaternion, obviating the quaternion complex field, which is a spin operator within a real vector space R^3 that generalizes the cross product (to a cross product plus a scaled vector addition of the operand similarly as the cosine [**cosk**] in the k -hat exponential) and corresponds to that for k -cross, j -cross, and i -cross on the three mutually perpendicular axes

$$\bar{x}(\cdot) = x_0 + x_1 \hat{i} \times (\cdot) + x_2 \hat{j} \times (\cdot) + x_3 \hat{k} \times (\cdot) = (x_0 + x_1 \hat{i} \times + x_2 \hat{j} \times + x_3 \hat{k} \times)(\cdot)$$

If $\bar{x} = x_1 \hat{i} + x_2 \hat{j} + x_3 \hat{k}$ is instead an angular position vector, and hence a vector basis, this can be written as

$$x_0 + \bar{x} \times (\cdot) = (x_0 + \bar{x} \times)(\cdot)$$

which is an operator and operates on the R^3 basis variables i -hat, j -hat, and k -hat, and any linear combination of the basis variables. This will make for more sophisticated and competent mathematics especially for spatial dynamics. The diadic vector basis expansion into the matrix basis is

$$(x_0 + \vec{x} \times)(\cdot) = \begin{pmatrix} x_0 & -x_3 & x_2 \\ x_3 & x_0 & -x_1 \\ -x_2 & x_1 & x_0 \end{pmatrix} (\cdot)$$

which is hence the standard cross product plus x_0 placed on the main diagonal. This is alternatively obtained diadically with the component operators of $\hat{i} \times$, $\hat{j} \times$, and $\hat{k} \times$

$$x_0 + x_1 \hat{i} \times + x_2 \hat{j} \times + x_3 \hat{k} \times = x_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + x_1 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} x_0 & -x_3 & x_2 \\ x_3 & x_0 & -x_1 \\ -x_2 & x_1 & x_0 \end{pmatrix}$$

This then makes possible a set of new definitions. After a few days thought, I was going to try to update all of the vector identities that involve the cross-product [vector of basis curls] but I am now not going to attempt that yet. Here are a few of the definitions that I think would be most important, starting with those from mechanics and then vector calculus and the Maxwell equations.

Mechanics

These are pre-existing functions from mechanics developed by Euler (Hamilton was about 50 years later) notably $D + \omega \times$ which pre-existed before quaternions (for Hamilton's attempt to extend complex numbers which have never quite made it into common mathematics, unlike complex numbers which are ubiquitous) but here (for this mechanics section) the mechanics functions are slightly renamed as quaternion, and so retaining and updating this very appropriate terminology for an entirely real R^3 basis, rather than a 4-vector, given that it is equivalent to the standard cross product. Replacing the x_1, x_2, x_3 with angular velocities $\omega_1, \omega_2, \omega_3$

Diadic vector basis expansion to matrix basis: the angular velocities (curls) populate the anti-diagonals of the matrix basis (as usual for a cross-product and curl) while the total time derivatives D run down the main diagonal, making it positive definite. I have always thought that the main diagonal should be for divergences but this has not been possible because of the fact that divergence is scalar. This should make integration easier and not have to rely on Stokes theorem and hence not have to have a closed contour to evaluate in integral form. Note again that this definition means that **(1)** quaternions are not complex while **(2)** the diagonal of D are eigenvalues.

Velocity quaternion for a rotating reference frame (an updated name for the classical Engineering mechanics operator $D + \vec{\omega} \times$)

$$D + \omega_1 \hat{i} \times (\cdot) + \omega_2 \hat{j} \times (\cdot) + \omega_3 \hat{k} \times (\cdot) = (D + \omega_1 \hat{i} \times + \omega_2 \hat{j} \times + \omega_3 \hat{k} \times)(\cdot)$$

where velocity quaternion is the velocity derivative in a rotating reference frame, i.e., the standard time derivative plus the cross product with angular velocities ω_1, ω_2 , and ω_3 . With a diadic expansion into a matrix basis

$$(D + \omega_1 \hat{i} \times + \omega_2 \hat{j} \times + \omega_3 \hat{k} \times)(\cdot) = \begin{pmatrix} D & -\omega_3 & \omega_2 \\ \omega_3 & D & -\omega_1 \\ -\omega_2 & \omega_1 & D \end{pmatrix}(\cdot)$$

where $D + \omega_1 \hat{i} \times + \omega_2 \hat{j} \times + \omega_3 \hat{k} \times$ is this matrices minimum polynomial basis factorisation [as converse of the diadic expansion], that is, the minimum polynomial basis factorization of this matrix from that which it has been diadically expanded. So the inverse of this polynomial, here in each of the three basis curls, is that which row reduces this matrix to the identity and further to its inverse if squared. Alternatively, with the diadically expanded operators of $\hat{k} \times$, $\hat{j} \times$, and $\hat{i} \times$

$$D + \omega_1 \hat{k} \times + \omega_2 \hat{j} \times + \omega_3 \hat{i} \times = D \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \omega_1 \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \omega_2 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} + \omega_3 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} D & -\omega_1 & \omega_2 \\ \omega_1 & D & -\omega_3 \\ -\omega_2 & \omega_3 & D \end{pmatrix}$$

If $\vec{r}$ is the position vector $\vec{r} = \hat{i}x + \hat{j}y + \hat{k}z$ then if one compares the velocity vector in cylindrical polar coordinates

$$\begin{aligned} \vec{r} &= \hat{r}\dot{r} + \hat{\theta}r\dot{\theta} + \hat{z}\dot{z} \\ &= \hat{r}\dot{r} + \hat{\theta}r\omega + \hat{z}\dot{z} \end{aligned}$$

with that of the diadically expanded form of D plus curl then one can immediately see that the diagonal terms are the r -hat translational velocities and the single component non-diagonal is the θ -hat rotational velocity. Note that it is k -cross, j -cross, and i -cross that place the rotational velocities into the positions on the non-diagonals [in the case of plane-polars and cylindrical-polars, in terms of k -cross: the other non-diagonal positions determined by j -cross and i -cross left empty]. This demonstrates that the main diagonal entries are divergences and the non-diagonals are curls which, ultimately, should be what Helmholtz theorem states, that is, any vector field is the sum of a divergence and a curl. This theorem, as it stands, does not make sense [another vexing issue of mathematics] because you can't add apples to oranges, i.e., you can't add a scalar to a vector. This might be how one can sensibly make sense of such a theorem in an intuitive way with a practical way of applying it (shown in later documents by change of base).

Note for the velocity quaternion when the coefficient of $j = \hat{j} \times$ is zero (and $\omega_3 = \kappa$ is curvature and $\omega_1 = \tau$ is torsion), these are Frenet-Serret equations for **tangential**, **normal**, and **binormal** coordinates (T , N , and B) when expressed as a function of arc-length.

Acceleration quaternion for a rotating reference frame (an updated name for the classical mechanics operator $(D + \overline{\omega \times})^2$)

$$(D + \omega_1 \hat{i} \times (\cdot) + \omega_2 \hat{j} \times (\cdot) + \omega_3 \hat{k} \times (\cdot))^2 = (D + \omega_1 \hat{i} \times + \omega_2 \hat{j} \times + \omega_3 \hat{k} \times)^2 (\cdot)$$

When this is expanded out and multiplied by mass, it is the sum of the **translational**, **azimuthal**, **Coriolis**, and **centripetal** forces. These forces are not complex or imaginary. Coriolis force is that which generates cycloidal systems in planetary weather [which is definitely very real].

Harmonic acceleration quaternion for a rotating reference frame (new concept and name for $(D + \overline{\omega \times})(D - \overline{\omega \times})$ in an attempt to extend the notion of a conjugate pair of roots from that with $i = \hat{k} \times$ to that with $i = \hat{k} \times$, $j = \hat{j} \times$, and $k = \hat{i} \times$)

$$\left(D + \omega_1 \hat{i} \times (\cdot) + \omega_2 \hat{j} \times (\cdot) + \omega_3 \hat{k} \times (\cdot) \right) \left(D - \omega_1 \hat{i} \times (\cdot) - \omega_2 \hat{j} \times (\cdot) - \omega_3 \hat{k} \times (\cdot) \right) = \left(D + \omega_1 \hat{i} \times + \omega_2 \hat{j} \times + \omega_3 \hat{k} \times \right) \left(D - \omega_1 \hat{i} \times - \omega_2 \hat{j} \times - \omega_3 \hat{k} \times \right) (\cdot)$$

Noting that this leads to considerable simplification of the diadically vector expanded transition matrix of that of the regular (non-conjugate) acceleration quaternion.

Vector Identities

There are three vector identities that involve the cross product (divergence of cross, curl of scalar product, and curl of cross)

$$\nabla \bullet (\bar{A} \times \bar{B}) \qquad \nabla \times (\phi \bar{A}) \qquad \nabla \times (\bar{A} \times \bar{B})$$

and the leading vector or scalar ($\bar{A}$ or ϕ) can be del (nabla, ∇) so that there are three more identities of then second order that involve the cross product (divergence of curl, curl of gradient, and curl of curl)

$$\nabla \bullet (\nabla \times \bar{B}) \qquad \nabla \times (\nabla \bar{A}) \qquad \nabla \times (\nabla \times \bar{B})$$

the first two of the second order identities being the null identities, which has been very vexing (probably to others also), and the last which is the Laplacian plus interaction terms. The Laplacian is very important, in my opinion, because it represents Gauss curvature, most importantly, the **Gauss distributed curvature** of the **Maxwell basis** (postulated by Newton, strongest advocate and proponent by Maxwell, searched for by Michealson and Morley [not discredited by Relativity], observed by Penzias and Wilson [cosmic microwave background] with the same properties and dynamics of thermodynamic heat Q).

I had intended for each of these to be improved for more sophistication, most notably time dynamics, each with the concept of quaternion velocity (standard derivative plus cross product) and quaternion curl (time partial added to nabla cross product) on R^3 . A main purpose of this is to update Maxwell's equations for more sophisticated time dynamics (without any relativistic funny business or nonsense which, as a personal opinion, wrecked my experience learning classical physics [PHYS2113]: relativity is not classical physics).

Vector calculus

These are new vector calculus operators and identities which extend **standard spatial curl**, which extends cross product to spatial derivatives, to **quaternion curl** which adds the time partial to the **nabla cross product** (nabla-cross in analogy to k -cross). These are similar to $D + \omega \times$ but with the angular velocity vector ω replaced by nabla (i.e., standard curl, $\omega \times$ replace by $\nabla \times$) and hence the time derivative that populates the main diagonal is a partial rather than total derivative. So if standard spatial curl is $\nabla \times$ then **time dynamic quaternion curl** is

$$\text{quaternion curl (definition)} \qquad \frac{\partial}{\partial t} + \frac{\partial}{\partial x} \hat{i} \times (\cdot) + \frac{\partial}{\partial y} \hat{j} \times (\cdot) + \frac{\partial}{\partial z} \hat{k} \times (\cdot) = \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x} \hat{i} \times + \frac{\partial}{\partial y} \hat{j} \times + \frac{\partial}{\partial z} \hat{k} \times \right) (\cdot)$$

$$\text{with the abbreviated notation} \qquad \partial_t + \partial_x \hat{i} \times (\cdot) + \partial_y \hat{j} \times (\cdot) + \partial_z \hat{k} \times (\cdot) = \left(\partial_t + \partial_x \hat{i} \times + \partial_y \hat{j} \times + \partial_z \hat{k} \times \right) (\cdot)$$

or abbreviated further to $\partial_t + \nabla \times = (\partial_t + \nabla \times)(\cdot)$ where the partials operate on the scalars and the k -cross, j -cross, and i -cross operate on basis variables. This is probably why 4-vectors have been found in certain circumstances to work, which if anything is not because of the ‘curvature of space time’, rather is the Gauss curvature (Theorema Egregium) of the Riemann sphere and hence Gauss distributed Maxwell basis, with Euler characteristic of the Gauss curvature of 2 similarly as (isometric with) the Platonic and Archimedean solids

$$2\pi\chi = \iint \kappa dA = \iint (1/R^2) dA = 4\pi R^2 / R^2 = 4\pi \quad \text{for a sphere, which is independent of bending and}$$

$$\chi = (V + F) - E = 2 \quad 2\pi\chi = 2\pi((V + F) - E) = 4\pi \quad \text{for Platonic and Archimedean solids, isometric with the sphere.}$$

Note that all dynamics occurs in **spatial coordinates** (x, y, z) and the **parameter of dynamics** is time t . Also, because of the $1/R^2$ laws, everything is curved. Hence, a curvature of space-time does not imply relativity. Whatever Einstein saw in the patent applications at the patent office and was not up front about (like the photo-electric effect discovered by Leonard) has not been explained properly [which is why people sound like they are talking nonsense just like risk-neutral valuation]. Einstein and Planck tried to give themselves credit for all of classical mechanics because of the failure to detect the ether [and forgetting to stress or in any way mention that discretization of the $1/R^2$ laws was Balmer’s mathematical 1885 discovery of the $1/n^2$ law for the emission lines of hydrogen]. It is radiation, no big surprise there. But radiation had not, as per x-rays and gamma rays, yet been discovered when Maxwell was searching for a basis. Hence the best of what was available at the time as an explanation for something that cannot be seen but can propagate waves was ether. The basis, which the ether theory was looking for, by its nature is curved [because, for example, all matter attracts all other matter isotropically per unit of nucleon]. This is the harmony of the spheres [in a dot-matrix Universe of basis components of electromagnetic heat Q].

The diadic expansion of the vector basis to a matrix basis then results in

$$(\partial_t + \partial_x \hat{i} \times + \partial_y \hat{j} \times + \partial_z \hat{k} \times)(\cdot) = \begin{pmatrix} \partial_t & -\partial_z & \partial_y \\ \partial_z & \partial_t & -\partial_x \\ -\partial_y & \partial_x & \partial_t \end{pmatrix}(\cdot)$$

Divergence of curl (div-curl) and the curl of gradient (curl-grad) were hence previously null, that is, the null scalar (the zero scalar) and the null vector (the zero vector). **Quaternion curl** means that they are no longer null (but still are with **standard spatial curl**) and essentially return the time dynamics. Hence, these two operators can then be used to separate out the time dynamics from the spatial dynamics when defined in this way. Note also that the time partial could also instead be a total (material) derivative which would expand spatial time derivatives also onto the main diagonal. Time will tell what works best. The curl of the gradient is the time partial of the gradient which is how the gradient changes with respect to time locally, whereby the convective terms are zeroed by the standard curl component (leaving the local time dynamics, as per fluid flow). Here are two examples where the extension might be relevant to Maxwell’s equations: Divergence of quaternion curl of the electric field E

$$\epsilon_0 \nabla \cdot (\partial_t + \nabla \times) E = \epsilon_0 \nabla \cdot \frac{\partial E}{\partial t} = \frac{\partial}{\partial t} (\epsilon_0 \nabla \cdot E) = \frac{\partial \rho}{\partial t}$$

which says that the divergence of the quaternion curl ($\partial_t + \nabla \times$) of the electric field ($\mathbf{E}$) equals charge density (ρ) accumulation/dissipation (in Coulombs per second) by the Gauss divergence theorem ($\epsilon_0 \nabla \cdot \mathbf{E} = \rho$). This may, for example, be the physics behind chemical electrode potentials [and hence electro-chemical cells, i.e., batteries]. For the magnetic field, the divergence remains null

$$\nabla \bullet (\partial_t + \nabla \times) \mathbf{H} = \nabla \bullet \frac{\partial \mathbf{H}}{\partial t} = \frac{\partial}{\partial t} (\nabla \bullet \mathbf{H}) = \frac{\partial}{\partial t} (0) = 0$$

given that div of curl is the zero scalar. From this nullity, charge density dissipation can be equated to divergence of current density for the continuity equation and hence to the standard exponential decay of charge as per standard electromagnetic theory [which then corresponds to circuit theory].

It has been made very clear that it is overwhelmingly important to distinguish between row and column vectors and that both forms of vector are equally important, for example, in diadic expansions. So here is one other major opportunity for an improvement to Maxwell's equations for additional sophistication, defining a row nabla and a column nabla: nabla and primed nabla (or visa versa). The scalar divergence would then be the inner product (as per standard divergence) and a newly defined matrix-type divergence the outer product. The outer product of a column vector nabla with a row vector $\mathbf{E}$ or $\mathbf{H}$ basis then takes the form of a Jacobian which applies to transforms of systems, the determinant for which being the scale factor of the transformation. In the context of electromagnetism, then, this would imply that the $\mathbf{E}$ and $\mathbf{H}$ fields are essentially coordinate transforms of the Cartesian coordinate system. In other words, this would suggest that the $\mathbf{E}$ and $\mathbf{H}$ fields are coordinate systems, coordinate systems obtained from transforming the rectangular coordinate system of R^3 . I have had suspicions that the $\mathbf{E}$ and $\mathbf{H}$ fields are actual coordinate systems, in the same sort of a way that I became convinced that complex numbers are the basis variables of R^3 (which they are once appended with $\times$).

A basis for R^3 generates R^3 so is essentially the coordinate system itself. If the Universal basis function is electromagnetic (which it undoubtedly is) then this implies that electromagnetism is a coordinate system [since it is also a basis].

Time dynamics is a shortcoming of the Maxwell equations and they might be getting a major overhaul so as to be applicable for the quantum mechanics of electron orbitals. Most particularly, the divergence of the Gauss equations for the electric and magnetic fields (the two divergence equations) need to have **(1)** explicit time dynamics and **(2)** be workable into vectors (which scalar divergences currently are not). These would then correspond to the first two quantum numbers, those of **shell** (n) and **orbital** (l). The two curl equations (Faraday and Ampere-Maxwell) would then correspond to the two spin quantum numbers, the **magnetic quantum number** (m_l) and **spin** (m_s). Time dependent Schrodinger's equation (noting that it involves k -cross), which is the heat equation, will also be getting an overhaul to correspond with n and l (**shell** and **orbital**) for a full Gauss density [obviating having to square the wavefunction which will then be just an approximation to the full density]. Everything is diffusion calculus (calculus extended from R [determinism] to S [entropy]) in an electromagnetic basis of heat. Hence the beginnings and everything ever since [Newton's Law of Cooling]. The Universe is a system of calculus in an electromagnetic basis of heat (one very large spring constant).

Appendix: Cauchy-Riemann Equations (2025/2026 Southern Hemisphere Summer)

1. Cauchy-Riemann equations; Complex Plane and Vector Plane equivalence with $i = \hat{k} \times$

The complex plane is equivalent to the R^3 vector volume with the rotation axis dimension latent so, for clarity, here $z = x + i y$ on the R^2 xy-vector plane is considered with the 'complex number' z parameterized as a circle ($\hat{k}$ -basis exponential) with $x = \cos \theta$ and $y = \sin \theta$.

This form of the complex number $z = x + i y$ is a generic circle (and rotation matrix) as it is determined by i . The form of i , and the partition into the even x component and conjugate odd y component determines its nature. For instance, here $i = k \times$ but it could also be $i = (k \times)^2$ (double cross exponential) or $i \times j \times + j \times i \times$ (hyperbolic exponential), $i \times j \times$ (nilpotent exponential) or $i \times j \times k \times$ (triple cross quaternion exponential) in which case the even x and odd y components would be parameterized differently, **coskk** and **sinkk**, or **coshk** or **sinhk**, or **cosij** or **sinij**, or **cosijk** and **sinijk**. When it is $i = k \times$ and the components are scalar reals, z is circular and so $x = \text{cosk}$ and $y = \text{sink}$ are the appropriate even and odd functions for the even and odd axes of R^2 once the basis exponential is mapped to the vector plane by applying it to a right handed zero phase basis. This is why the Cauchy-Riemann equations hold, because y is a phase shift of x in implicit circular coordinates.

The Cauchy-Riemann equations are discussed now with the objective of showing that functions of circular z (that of $i = \hat{k} \times$) are harmonic as a function of circular angle θ .

Showing that a complex function satisfies the Cauchy-Riemann equations demonstrates that a function is differentiable on the complex plane, which is exactly equivalent one for one on the xy-vector plane with $i = \hat{k} \times$. It is a fact that a circle is differentiable so the Cauchy-Riemann equations must hold. Similarly for a generalized circle consisting of these circles at harmonic multiples of the base circle frequency (the fundamental); it is clearly differentiable, so these equations must hold. It is shown that these equations do hold, and by the same methods taken from first order to second, it is then shown that they satisfy Laplace's equation. This then demonstrates that generalized circles are harmonic, where a generalized circle is defined as a cosine series on the even x axis and the corresponding sine series on the odd y axis, arising from a series of basis exponentials at integer multiples of the base frequency for the rotation axis $i = k \times$, after being applied to the zero phase basis. A generalized circle is then a Fourier cosine series for the even x -axis and the conjugate sine series for the odd axis. By maintaining conjugate pairs, each term is then maintained as a circle (without eccentricity) at that frequency. These functions are specifically for differentiability with respect to angle θ , and θ only. It is not a differential function with respect to radius r specifically for the criterion as a generalized circle (although the Cauchy Riemann equations also hold for that with dynamic r but here it is specifically for fixed radii and with matched odd and even conjugate functions).

In complex notation

$$z = x + iy$$

$$f(z) = u(x, y) + i v(x, y)$$

In vector notation ($i = \hat{k} \times$ and zero phase standard basis)

$$\vec{r} \hat{i} = \hat{i}x + \hat{j}y$$

$$\vec{r} = x + y \hat{k} \times$$

$$f(\vec{r}) \hat{i} = \hat{i}u(x, y) + \hat{j}v(x, y) \text{ with } \hat{k} \text{ latent}$$

Cauchy-Riemann Differentiation (on complex plane)Differentiating partially with respect to x

$$\begin{aligned}
 f'(z) &= \frac{\partial}{\partial x} f(z) \\
 &= \frac{\partial}{\partial x} (u(x, y) + iv(x, y)) \\
 &= \frac{\partial}{\partial x} u(x, y) + i \frac{\partial}{\partial x} v(x, y) \\
 &= u_x + iv_y
 \end{aligned}$$

Differentiating partially with respect to $i y$

$$\begin{aligned}
 f'(z) &= \frac{\partial}{\partial(iy)} f(z) \\
 &= (-i) \frac{\partial}{\partial y} f(z) \\
 &= (-i) \frac{\partial}{\partial y} (u(x, y) + iv(x, y)) \\
 &= (-i) \frac{\partial}{\partial y} u(x, y) - i^2 \frac{\partial}{\partial y} v(x, y) \\
 &= \frac{\partial}{\partial y} v(x, y) - i \frac{\partial}{\partial y} u(x, y) \\
 &= v_y - iu_y
 \end{aligned}$$

Equating the equivalent partials

$$f'(z) = \frac{\partial}{\partial x} f(z) \text{ and } f'(z) = \frac{\partial}{\partial(iy)} f(z) \text{ are equal}$$

$$\begin{aligned}
 \frac{\partial}{\partial x} u(x, y) + i \frac{\partial}{\partial x} v(x, y) &= \frac{\partial}{\partial y} v(x, y) - i \frac{\partial}{\partial y} u(x, y) \\
 u_x + iv_x &= v_y - iu_y
 \end{aligned}$$

and equating real and imaginary parts

$$\frac{\partial}{\partial x} u(x, y) = \frac{\partial}{\partial y} v(x, y) \quad \text{and} \quad \frac{\partial}{\partial x} v(x, y) = -\frac{\partial}{\partial y} u(x, y)$$

$$\text{i.e., } u_x(x, y) = v_y(x, y) \quad \text{and} \quad v_x(x, y) = -u_y(x, y)$$

and most easily remembered for exams as $u_x = v_y$ and $u_y = -v_x$.**Cauchy-Riemann Differentiation (on vector plane)**Differentiating partially with respect to x

$$\begin{aligned}
 f'(\vec{r}) &= \frac{\partial}{\partial x} f(\vec{r}) \\
 &= \frac{\partial}{\partial x} (\hat{i}u(x, y) + \hat{j}v(x, y)) \\
 &= \hat{i} \frac{\partial}{\partial x} u(x, y) + \hat{j} \frac{\partial}{\partial x} v(x, y) \\
 &= \hat{i}u_x + \hat{j}v_y
 \end{aligned}$$

Differentiating partially with respect to $y \hat{k} \times$

$$\begin{aligned}
 f'(\vec{r}) &= \frac{\partial}{\partial(y \hat{k} \times)} f(\vec{r}) \\
 &= (-\hat{k} \times) \frac{\partial}{\partial y} f(\vec{r}) \\
 &= (-\hat{k} \times) \frac{\partial}{\partial y} (\hat{i}u(x, y) + \hat{j}v(x, y)) \\
 &= (-\hat{k} \times \hat{i}) \frac{\partial}{\partial y} u(x, y) + (-\hat{k} \times \hat{j}) \frac{\partial}{\partial y} v(x, y) \\
 &= \hat{i} \frac{\partial}{\partial y} v(x, y) - \hat{j} \frac{\partial}{\partial y} u(x, y) \\
 &= \hat{i}v_y - \hat{j}u_y
 \end{aligned}$$

Equating the equivalent phase shifted partials

$$f'(\vec{r}) = \frac{\partial}{\partial x} f(\vec{r}) \text{ and } f'(\vec{r}) = \frac{\partial}{\partial(y \hat{k} \times)} f(\vec{r}) \text{ are equal}$$

$$\begin{aligned}
 \hat{i} \frac{\partial}{\partial x} u(x, y) + \hat{j} \frac{\partial}{\partial x} v(x, y) &= \hat{i} \frac{\partial}{\partial y} v(x, y) - \hat{j} \frac{\partial}{\partial y} u(x, y) \\
 \hat{i}u_x + \hat{j}v_x &= \hat{i}v_y - \hat{j}u_y
 \end{aligned}$$

and equating i -hat and j -hat components

$$\frac{\partial}{\partial x} u(x, y) = \frac{\partial}{\partial y} v(x, y) \quad \text{and} \quad \frac{\partial}{\partial x} v(x, y) = -\frac{\partial}{\partial y} u(x, y)$$

$$\text{i.e., } u_x(x, y) = v_y(x, y) \quad \text{and} \quad v_x(x, y) = -u_y(x, y)$$

which is exactly the same scalar equations as that in complex plane notation, with the one caveat of division by singular $\hat{k} \times$

Hence side by side shows unequivocally that the Cauchy-Riemann equations in complex notation hold exactly equivalent for vector algebra equations with $i = \hat{k} \times$ on the zero phase basis xy-vector plane. This complex plane mathematics works exactly the same with $i = \hat{k} \times$, and then applies one-for-one with the R^2 vector plane (R^3 with $\hat{k}$ axis latent) once applied to the standard basis, ordinarily the zero phase standard basis. In some ways it seems somewhat trivial, but it is not. For example, all of the complex integration is possible without having to throw away the complex part, and also possible to do these integrations for volumes in R^3 once the relevant rotation eigenvalues are taken into account with **expk**, **expj**, and **expi**. All of this maths is applicable to each of the three real vector planes of R^3 (rather than just one plane in some detached esoteric non-existent space). It also makes it rock solid in how it works which means better intuition and less scope for screw-ups which was the motivation, and ongoing theme, for defining $i = \hat{k} \times$. It can also be demonstrated in the diadically expanded matrix basis.

The transfer function in the diadically expanded matrix basis is

$$\mathbf{f}(x, y; \hat{k} \times) = \mathbf{u}(x, y) + \mathbf{v}(x, y) \hat{k} \times = \begin{pmatrix} u(x, y) & 0 & 0 \\ 0 & u(x, y) & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -v(x, y) & 0 \\ v(x, y) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} u(x, y) & -v(x, y) & 0 \\ v(x, y) & u(x, y) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(1) Differentiating partially with respect to x

$$\frac{\partial}{\partial x} \mathbf{f}(x, y; \hat{k} \times) = \frac{\partial}{\partial x} \mathbf{u}(x, y) + \frac{\partial}{\partial x} \mathbf{v}(x, y) \hat{k} \times = \frac{\partial}{\partial x} \begin{pmatrix} u(x, y) & -v(x, y) & 0 \\ v(x, y) & u(x, y) & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \partial u(x, y) / \partial x & -\partial v(x, y) / \partial x & 0 \\ \partial v(x, y) / \partial x & \partial u(x, y) / \partial x & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} u_x & -v_x & 0 \\ v_x & u_x & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(2) Differentiating partially with respect to $y \hat{k} \times$

$$\frac{\partial}{\partial y \hat{k} \times} \mathbf{f}(x, y; \hat{k} \times) = -\hat{k} \times \left(\frac{\partial}{\partial y} \mathbf{u}(x, y) + \frac{\partial}{\partial y} \mathbf{v}(x, y) \hat{k} \times \right) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \partial u(x, y) / \partial y & -\partial v(x, y) / \partial y & 0 \\ \partial v(x, y) / \partial y & \partial u(x, y) / \partial y & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} v_y & u_y & 0 \\ -u_y & v_y & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and equating $\frac{\partial}{\partial x} \mathbf{f}(x, y; \hat{k} \times)$ with $\frac{\partial}{\partial y \hat{k} \times} \mathbf{f}(x, y; \hat{k} \times)$ because the latter is a phase shift of the other before differentiation, so each of the partial derivatives are also equal

$$\begin{aligned} \frac{\partial}{\partial x} \mathbf{f}(x, y; \hat{k} \times) &= \frac{\partial}{\partial y \hat{k} \times} \mathbf{f}(x, y; \hat{k} \times) \\ \begin{pmatrix} \partial u(x, y) / \partial x & -\partial v(x, y) / \partial x & 0 \\ \partial v(x, y) / \partial x & \partial u(x, y) / \partial x & 0 \\ 0 & 0 & 0 \end{pmatrix} &= \begin{pmatrix} \partial v(x, y) / \partial y & \partial u(x, y) / \partial y & 0 \\ -\partial u(x, y) / \partial y & \partial v(x, y) / \partial y & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} u_x & -v_x & 0 \\ v_x & u_x & 0 \\ 0 & 0 & 0 \end{pmatrix} &= \begin{pmatrix} v_y & u_y & 0 \\ -u_y & v_y & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

Either of the diagonal terms give the first of the Cauchy-Riemann equations $u_x(x, y) = v_y(x, y)$ and either of the anti-diagonals give the second $v_x(x, y) = -u_y(x, y)$. Similarly, for Laplace's equation

$$(1) \frac{\partial^2}{\partial x^2} \mathbf{f}(x, y; \hat{k} \times) = \frac{\partial^2}{\partial x^2} \mathbf{u}(x, y) + \frac{\partial^2}{\partial x^2} \mathbf{v}(x, y) \hat{k} \times = \frac{\partial}{\partial x} \begin{pmatrix} u_x & -v_x & 0 \\ v_x & u_x & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} u_{xx} & -v_{xx} & 0 \\ v_{xx} & u_{xx} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$(2) \left(\frac{\partial}{\partial y \hat{k} \times} \right)^2 \mathbf{f}(x, y; \hat{k} \times) = (\hat{k} \times)^2 \left(\frac{\partial^2}{\partial y^2} \mathbf{u}(x, y) + \frac{\partial^2}{\partial y^2} \mathbf{v}(x, y) \hat{k} \times \right) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{\partial}{\partial y} \begin{pmatrix} v_y & u_y & 0 \\ -u_y & v_y & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -u_{yy} & v_{yy} & 0 \\ -v_{yy} & -u_{yy} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and equating the second order partials similarly for first order $\frac{\partial^2}{\partial x^2} \mathbf{f}(x, y; \hat{k} \times) = \left(\frac{\partial}{\partial y \hat{k} \times} \right)^2 \mathbf{f}(x, y; \hat{k} \times)$, that is

$$\begin{pmatrix} u_{xx} & -v_{xx} & 0 \\ v_{xx} & u_{xx} & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -u_{yy} & v_{yy} & 0 \\ -v_{yy} & -u_{yy} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Equating either of the diagonals for the even part $u_{xx} = -u_{yy}$ i.e., $u_{xx} + u_{yy} = 0$ and either of the anti-diagonals for the conjugate odd part $-v_{xx} = v_{yy}$ i.e., $v_{xx} + v_{yy} = 0$. Hence

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

and similarly

$$\frac{\partial^2 \mathbf{u}}{\partial x^2} + \frac{\partial^2 \mathbf{u}}{\partial y^2} = 0 \qquad \frac{\partial^2 \mathbf{v}}{\partial x^2} + \frac{\partial^2 \mathbf{v}}{\partial y^2} = 0$$

Therefore

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \mathbf{f}(x, y; \hat{k} \times) = 0$$

Note importantly that $\mathbf{u}$ and $\mathbf{v}$ are operators such as **cosk** and **sink**, as components of **expk** for instance, whilst $x = \cos\theta$ and $y = \sin\theta$ are scalar functions within these diadically expanded operators. This makes these functions those that comprise the inner workings of the dynamics of a transfer function (such as **expk**). Despite conflicting x and y notation for position vectors, the scalars $x = \cos\theta$ and $y = \sin\theta$ are not positions for the coordinate vector upon which these operators operate, i.e., $x = \cos\theta$ and $y = \sin\theta$ are not the scalar components of a coordinate position vector that is mapped to a forward transition by a transfer function.

Refresher: note the functions $x = \cos\theta$, $y = \sin\theta$, $2xy = \sin 2\theta$, $x^2 - y^2 = \cos 2\theta$, $x^3 - 3xy^2 = \cos 3\theta$, and $3x^2y - y^3 = \sin 3\theta$.

Example $\mathbf{f}(x, y; \hat{k} \times) = \mathbf{expk} \, 3\theta = \begin{pmatrix} \cos 3\theta & -\sin 3\theta & 0 \\ \sin \theta & \cos 3\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} x^3 - 3xy^2 & -3x^2y + y^3 & 0 \\ 3x^2y - y^3 & x^3 - 3xy^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ where $x = \cos \theta$ and $y = \sin \theta$

so here the even function is $u(x, y) = \cos 3\theta = x^3 - 3xy^2$ and the conjugate odd function is $v(x, y) = \sin 3\theta = 3x^2y - y^3$.

$$u_x = 3x^2 - 3y^2 \quad v_y = 3x^2 - 3y^2 \quad \text{hence} \quad u_x = v_y = 3x^2 - 3y^2$$

$$u_y = -6xy \quad v_x = 6xy \quad \text{hence} \quad v_x = -u_y = 6xy$$

Therefore, the Cauchy-Riemann equations $u_x = v_y$ and $v_x = -u_y$ are satisfied and the transfer function is analytic (as we know anyway, but here shows why these equations hold). The relevant second order partial derivatives are

$$u_{xx} = 6x \quad u_{yy} = -6x \quad \text{hence} \quad u_{xx} = -u_{yy} = 6x \quad \text{i.e., } u_{xx} + u_{yy} = 0$$

$$v_{xx} = 6y \quad v_{yy} = -6y \quad \text{hence} \quad v_{xx} = -v_{yy} = 6y \quad \text{i.e., } v_{xx} + v_{yy} = 0$$

so by u and v are harmonic and therefore so is f , i.e., the rotation matrix at the second overtone **expk** 3θ is harmonic. In diadically expanded matrix form, the partial derivatives are

$$\frac{\partial}{\partial x} \mathbf{f}(x, y; \hat{k} \times) = \frac{\partial}{\partial x} \begin{pmatrix} x^3 - 3xy^2 & -3x^2y + y^3 & 0 \\ 3x^2y - y^3 & x^3 - 3xy^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 3x^2 - 3y^2 & -6xy & 0 \\ 6xy & 3x^2 - 3y^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$(-\hat{k} \times) \frac{\partial}{\partial y} \mathbf{f}(x, y; \hat{k} \times) = (-\hat{k} \times) \frac{\partial}{\partial y} \begin{pmatrix} x^3 - 3xy^2 & -3x^2y + y^3 & 0 \\ 3x^2y - y^3 & x^3 - 3xy^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -6xy & -3x^2 + 3y^3 & 0 \\ 3x^2 - 3y^2 & -6xy & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 3x^2 - 3y^2 & -6xy & 0 \\ 6xy & 3x^2 - 3y^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Therefore $\frac{\partial}{\partial x} \mathbf{f}(x, y; \hat{k} \times) = (-\hat{k} \times) \frac{\partial}{\partial y} \mathbf{f}(x, y; \hat{k} \times)$ (all components are equal) and so $\mathbf{f}(x, y; \hat{k} \times) = \mathbf{expk} \, 3\theta$ is analytic.

$$\frac{\partial^2}{\partial x^2} \mathbf{f}(x, y; \hat{k} \times) = \frac{\partial}{\partial x} \begin{pmatrix} 3x^2 - 3y^2 & -6xy & 0 \\ 6xy & 3x^2 - 3y^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 6x & -6y & 0 \\ 6y & 6x & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\left((-\hat{k} \times) \frac{\partial}{\partial y} \right)^2 \mathbf{f}(x, y; \hat{k} \times) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{\partial}{\partial y} \begin{pmatrix} 3x^2 - 3y^2 & -6xy & 0 \\ 6xy & 3x^2 - 3y^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 6x & -6y & 0 \\ 6y & 6x & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

So $\left(\frac{\partial}{\partial x}\right)^2 \mathbf{f}(x, y; \hat{k} \times) = \left(-\hat{k} \times \frac{\partial}{\partial y}\right)^2 \mathbf{f}(x, y; \hat{k} \times)$ therefore

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \mathbf{f}(x, y; \hat{k} \times) = 0$$

and so $\mathbf{f} = \mathbf{expk} \ 3\theta$ is harmonic. Note that the actual transfer function is applied to the backward variables (initial conditions) for a forward motion (X, Y, Z) is

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \mathbf{f}(x, y; \hat{k} \times) \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = (\mathbf{expk} \ 3\theta) \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = \begin{pmatrix} \cos 3\theta & -\sin 3\theta & 0 \\ \sin \theta & \cos 3\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = \begin{pmatrix} x^3 - 3xy^2 & -3x^2y + y^3 & 0 \\ 3x^2y - y^3 & x^3 - 3xy^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$$

so x_0 and y_0 , are a different beast from $x = \cos \theta$ and $y = \sin \theta$ (despite being in the same equation and despite having conflicting notation). Just pointing it out; x and y are scalar features within the transfer function whilst x_0 and y_0 (and z_0) are the initial conditions of the variables that track the motion (those variables in the position vector that is operated on by the transfer function, here in this example, by **expk** 3θ).

This works out for all linear combinations of **expk** at integer multiples of the fundamental frequency ($\theta = \omega t$), that is, in terms of any linear combination of polynomials obtained from **expk** expressed in terms of $x = \cos \theta$ and $y = \sin \theta$. Sometimes there are strange functions for $\mathbf{u}$ and $\mathbf{v}$, such as $x = \cos \theta$ and $y = \sin \theta$ contained within e and within further sines and cosines, rather than just x and y polynomials; what is going on there? That is where there are transfer functions within transfer functions, that is, rotation (or hyperbolic or divergence/convergence) matrices within rotation matrices. It still all works out the same (subject to caveats of being overall harmonic for example) after expanding in terms of $x = \cos \theta$ and $y = \sin \theta$ for the above partial derivatives.

Form here, these equations will be shown to hold for the base exponential **expk**, and then for compositions of powers of the base circle, notably for generalized circles (which are power series/polynomials of **expk** and hence of rotation matrices which result in cosine Fourier series on the even axis and the corresponding conjugate sine Fourier series on the conjugate odd axis).

2. Cauchy-Riemann equations for the base circle, **expk**, of a generalized circle

2a. For the base circle, **expk**, of a generalized circle $e^{\theta \hat{k} \times} = \mathbf{cosk} \theta + (\mathbf{sink} \theta) \hat{k} \times$, in the i -hat direction $e^{\theta \hat{k} \times} \hat{i} = \hat{i} \cos \theta + \hat{j} \sin \theta$ so

$$\begin{aligned} \frac{\partial}{\partial \theta} (e^{\theta \hat{k} \times} \hat{i}) &= \frac{\partial}{\partial \theta} (\hat{i} \cos \theta + \hat{j} \sin \theta) \\ &= \hat{i}(-\sin \theta) + \hat{j} \cos \theta \\ &= e^{\theta \hat{k} \times} \hat{j} \end{aligned}$$

which is saying everything that the Riemann-Cauchy equations are saying, notably that **expk** is differentiable on the R^2 vector plane (as a subset of real R^3 rather than some other space or imaginary field). This holds because

$$\frac{\partial}{\partial \theta} (e^{\theta \hat{k} \times}) = (\hat{k} \times) e^{\theta \hat{k} \times} \quad \text{i.e.,} \quad \frac{\partial}{\partial \theta} (e^{\theta \hat{k} \times}) \hat{i} = (\hat{k} \times) e^{\theta \hat{k} \times} \hat{i} = e^{\theta \hat{k} \times} (\hat{k} \times) \hat{i} = e^{\theta \hat{k} \times} \hat{j}$$

Parameterizing the base circle as per complex theory, $x = \cos \theta$ and $y = \sin \theta$

Cauchy-Riemann Differentiation

$$e^{i\theta} = \cos \theta + i \sin \theta$$

$$e^{i\theta} = x + iy$$

$$\text{i.e., } z = e^{i\theta} \quad x = \cos \theta \quad y = \sin \theta$$

Vector Plane Differentiation

$$e^{\theta \hat{k} \times} \hat{i} = \hat{i} \cos \theta + \hat{j} \sin \theta$$

$$e^{\theta \hat{k} \times} \hat{i} = \hat{i} x + \hat{j} y$$

$$\text{i.e., } \vec{r} = e^{\theta \hat{k} \times} \hat{i} \quad x = \cos \theta \quad y = \sin \theta$$

First Cauchy-Riemann equation

$$\frac{\partial}{\partial x} e^{i\theta} = \frac{\partial}{\partial x} (x + iy) = \frac{\partial}{\partial x} x + i \frac{\partial}{\partial x} y = 1 + i0$$

$$\frac{\partial}{\partial y} e^{i\theta} = \frac{\partial}{\partial y} (x + iy) = \frac{\partial}{\partial y} x + i \frac{\partial}{\partial y} y = 0 + i1$$

$$\begin{aligned} \frac{\partial}{\partial(iy)} e^{i\theta} &= (-i) \frac{\partial}{\partial y} (e^{i\theta}) \\ &= (-i)(0 + i1) \\ &= (-i)0 + (-i^2)1 \\ &= 1 + i0 \\ &= \frac{\partial}{\partial x} e^{i\theta} \end{aligned}$$

$$\frac{\partial}{\partial x} (e^{\theta \hat{k} \times} \hat{i}) = \frac{\partial}{\partial x} (\hat{i} x + \hat{j} y) = \hat{i} \frac{\partial}{\partial x} x + \hat{j} \frac{\partial}{\partial x} y = \hat{i} 1 + \hat{j} 0$$

$$\frac{\partial}{\partial y} (e^{\theta \hat{k} \times} \hat{i}) = \frac{\partial}{\partial y} (\hat{i} x + \hat{j} y) = \hat{i} \frac{\partial}{\partial y} x + \hat{j} \frac{\partial}{\partial y} y = \hat{i} 0 + \hat{j} 1$$

$$\begin{aligned} \frac{\partial}{\partial(y \hat{k} \times)} (e^{\theta \hat{k} \times} \hat{i}) &= (-\hat{k} \times) \frac{\partial}{\partial y} (e^{\theta \hat{k} \times} \hat{i}) \\ &= (-\hat{k} \times) (\hat{i} 0 + \hat{j} 1) \\ &= (-\hat{k} \times \hat{i}) 0 + (-\hat{k} \times \hat{j}) 1 \\ &= \hat{i} 1 + \hat{j} 0 \\ &= \frac{\partial}{\partial x} (e^{\theta \hat{k} \times} \hat{i}) \end{aligned}$$

since the equality of these derivative holds (that for x and that for phase shifted y)

$$\frac{\partial}{\partial x} e^{i\theta} = \frac{\partial}{\partial(iy)} e^{i\theta} = 1 + i0$$

the real and imaginary components of the derivatives are equivalent, which are the Cauchy-Riemann equations.

since the equality of these derivative holds (that for x and that for phase shifted y)

$$\frac{\partial}{\partial x} (e^{\theta \hat{k} \times} \hat{i}) = \frac{\partial}{\partial(y \hat{k} \times)} (e^{\theta \hat{k} \times} \hat{i}) = \hat{i} 1 + \hat{j} 0$$

the i -hat and j -hat components of the derivatives are equivalent.

$$u_x = v_y = 1 \quad \text{and} \quad v_x = -u_y = 0 \quad (\text{for } u = x \text{ and } v = y) \quad u_x = v_y = 1 \quad \text{and} \quad v_y = -u_x = 0 \quad (\text{for } u = x \text{ and } v = y)$$

which is semi-trivial for the base circle, $u(x, y) = x$ and $v(x, y) = y$, since the second Cauchy-Riemann equation is null ($v_y = -u_x = 0$) although stills shows that it is equivalent to that of the R^2 vector plane as a subset of R^3 (rather than some other esoteric imaginary field), one-for-one whilst also ensuring that it is differentiable. If the 90° phase shifted basis is applied (**expk** applied to j -hat rather than a zero phase to i -hat), then the second set of the Cauchy-Riemann equations hold ($v_y = -u_x = 1$) rather than first which is instead that which is null ($u_x = -v_y = 0$)

$$\frac{\partial}{\partial x} (e^{\theta \hat{k} \times \hat{j}}) = \frac{\partial}{\partial x} (-\hat{i}y + \hat{j}x) = -\hat{i} \frac{\partial}{\partial x} y + \hat{j} \frac{\partial}{\partial x} x = \hat{i}0 + \hat{j}1$$

$$\frac{\partial}{\partial y} (e^{\theta \hat{k} \times \hat{j}}) = \frac{\partial}{\partial y} (-\hat{i}y + \hat{j}x) = -\hat{i} \frac{\partial}{\partial y} y + \hat{j} \frac{\partial}{\partial y} x = \hat{i}(-1) + \hat{j}0$$

$$\begin{aligned} \frac{\partial}{\partial(y\hat{k} \times)} (e^{\theta \hat{k} \times \hat{j}}) &= (-\hat{k} \times) \frac{\partial}{\partial y} (e^{\theta \hat{k} \times \hat{j}}) \\ &= (-\hat{k} \times) (\hat{i}(-1) + \hat{j}0) \\ &= (\hat{k} \times \hat{i})1 + (-\hat{k} \times \hat{j})0 \\ &= \hat{i}0 + \hat{j}1 \\ &= \frac{\partial}{\partial x} (e^{\theta \hat{k} \times \hat{j}}) \end{aligned}$$

and hence $u_x = v_y = 0$ and $v_y = -u_x = 1$ (for $u = x$ and $v = y$)

Because of the one null relation of the two, the base circle at the fundamental frequency is sometimes not referred to as harmonic (it is clearly cyclical given the base frequency). It is the overtones that are considered harmonic for a generalized circle and hence for Fourier series, here cosine and conjugate sine on the respective even and odd conjugate axes.

2b. This derivative shows that differentiation with respect to the spatial even and odd variables is legitimate whilst holding the radius constant

$$\frac{d}{d\theta} = \frac{dx}{d\theta} \frac{\partial}{\partial x} + \frac{dy}{d\theta} \frac{\partial}{\partial y}$$

with $x = \cos \theta$, $y = \sin \theta$, $\frac{\partial x}{\partial \theta} = -\sin \theta$, and $\frac{\partial y}{\partial \theta} = \cos \theta$

vector algebra and calculus

complex algebra and calculus

$$\begin{aligned} \frac{d}{d\theta} (e^{\theta \hat{k} \times \hat{i}}) &= \frac{dx}{d\theta} \frac{\partial}{\partial x} (e^{\theta \hat{k} \times \hat{i}}) + \frac{dy}{d\theta} \frac{\partial}{\partial y} (e^{\theta \hat{k} \times \hat{i}}) \\ &= \frac{dx}{d\theta} \frac{\partial}{\partial x} (\hat{i}x + \hat{j}y) + \frac{dy}{d\theta} \frac{\partial}{\partial y} (\hat{i}x + \hat{j}y) \\ &= \left(\frac{d}{d\theta} \cos \theta \right) \frac{\partial}{\partial x} (\hat{i}x + \hat{j}y) + \left(\frac{d}{d\theta} \sin \theta \right) \frac{\partial}{\partial y} (\hat{i}x + \hat{j}y) \\ &= (-\sin \theta) (\hat{i}1 + \hat{j}0) + (\cos \theta) (\hat{i}0 + \hat{j}1) \\ &= (\hat{i}(-\sin \theta) + \hat{j}0) + (\hat{i}0 + \hat{j} \cos \theta) \\ &= \hat{i}(-\sin \theta) + \hat{j} \cos \theta \\ &= e^{\theta \hat{k} \times \hat{j}} \\ &= (\hat{k} \times) e^{\theta \hat{k} \times \hat{i}} \end{aligned}$$

$$\begin{aligned} \frac{d}{d\theta} (e^{i\theta}) &= \frac{dx}{d\theta} \frac{\partial}{\partial x} (e^{i\theta}) + \frac{dy}{d\theta} \frac{\partial}{\partial y} (e^{i\theta}) \\ &= \frac{dx}{d\theta} \frac{\partial}{\partial x} (x + iy) + \frac{dy}{d\theta} \frac{\partial}{\partial y} (x + iy) \\ &= \left(\frac{d}{d\theta} \cos \theta \right) \frac{\partial}{\partial x} (x + iy) + \left(\frac{d}{d\theta} \sin \theta \right) \frac{\partial}{\partial y} (x + iy) \\ &= (-\sin \theta) (1 + i0) + (\cos \theta) (0 + i1) \\ &= (-\sin \theta + i0) + (0 + i \cos \theta) \\ &= -\sin \theta + i \cos \theta \\ &= i (\cos \theta + i \sin \theta) \\ &= i e^{i\theta} \end{aligned}$$

as before. This is legitimate mathematics. All avenues show the same result; the Cauchy-Riemann equations, the equivalent R^2 vector calculus (as a subset of the real R^3 vector volume), and the total derivative all produce the same result. It is definitely okay to omit partials for the radius given that the radius is not changing for the base circle or those that comprise a harmonic composite (power series) of the base circle since the coefficients are fixed for tracing out the generalized circle. It is okay to do this since working with x and y is the same in standard vector calculus where these spatial variables are manipulated as ordinary algebraic variables and then only parameterized as polars (scalar sine and cosine) when it comes time to integrate.

2c. This can also be demonstrated in diadically expanded operators in a matrix basis, i.e., the Cauchy-Riemann equations also hold for the diadically expanded operators. Firstly, for the Cauchy-Riemann equations (2c) and then for the total derivatives (2d) to ensure it is all working correctly. The base circle **expk** is

$$e^{\theta \hat{k} \times} = \mathbf{cosk}\theta + (\mathbf{sink}\theta)\hat{k} \times = \begin{pmatrix} \cos\theta & 0 & 0 \\ 0 & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} \sin\theta & 0 & 0 \\ 0 & \sin\theta & 0 \\ 0 & 0 & \theta \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \cos\theta & 0 & 0 \\ 0 & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -\sin\theta & 0 \\ \sin\theta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and with $x = \cos\theta$ and $y = \sin\theta$

$$(1) \mathbf{u}(x, y) = \mathbf{u} = \mathbf{x} = \begin{pmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (2) \mathbf{v}(x, y)\hat{k} \times = \mathbf{v}\hat{k} \times = \mathbf{y}\hat{k} \times = \begin{pmatrix} 0 & -y & 0 \\ y & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

i.e.,

$$e^{\theta \hat{k} \times} = \mathbf{cosk}\theta + (\mathbf{sink}\theta)\hat{k} \times = \mathbf{x} + \mathbf{y}\hat{k} \times = \begin{pmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -y & 0 \\ y & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} x & -y & 0 \\ y & x & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

With derivatives $\frac{\partial x}{\partial \theta} = \cos\theta$ and $\frac{\partial y}{\partial \theta} = \sin\theta$, the partial derivatives for the even and odd components are

$$(1) \frac{\partial \mathbf{u}(x, y)}{\partial x} = \frac{\partial \mathbf{x}}{\partial x} = \frac{\partial}{\partial x} \begin{pmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$(2) (-\hat{k} \times) \frac{\partial \mathbf{v}(x, y)\hat{k} \times}{\partial y} = (-\hat{k} \times) \frac{\partial \mathbf{y}\hat{k} \times}{\partial y} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{\partial}{\partial y} \begin{pmatrix} 0 & -y & 0 \\ y & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Therefore, the first Cauchy-Riemann equation holds

$$\frac{\partial \mathbf{u}(x, y)}{\partial x} = \frac{\partial \mathbf{v}(x, y)}{\partial y} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \mathbf{u}_x = \mathbf{v}_y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

as for the vector basis, the Cauchy-Riemann equations also hold in the matrix basis, albeit for the base circle semi-degenerately - it holds only non-trivially for the first of the two equations, the second set hold because both derivatives are null. Both the relevant partial derivatives are null,

$$(1) \frac{\partial \mathbf{u}(x, y)}{\partial y} = \frac{\partial}{\partial y} \begin{pmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$(2) (-\hat{k} \times) \frac{\partial \mathbf{v}(x, y) \hat{k} \times}{\partial x} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{\partial}{\partial x} \begin{pmatrix} 0 & -y & 0 \\ y & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

so

$$\frac{\partial \mathbf{u}(x, y)}{\partial y} = -\frac{\partial \mathbf{v}(x, y)}{\partial x} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \mathbf{u}_y = -\mathbf{v}_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

2d. To show differentiating in the matrix basis holds for the total derivative, for which the two coupled equations describe, firstly for straight out differentiation with respect to θ and then as the sum of weighted partial derivatives. Noting that

$$e^{\theta \hat{k} \times} = \mathbf{cosk}\theta + (\mathbf{sink}\theta) \hat{k} \times = \begin{pmatrix} \cos\theta & 0 & 0 \\ 0 & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} \sin\theta & 0 & 0 \\ 0 & \sin\theta & 0 \\ 0 & 0 & \theta \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and the derivatives of the even and odd components with respect to θ are

$$\frac{d\mathbf{x}}{d\theta} = \frac{d}{d\theta} \mathbf{cosk}\theta = \frac{d}{d\theta} \begin{pmatrix} \cos\theta & 0 & 0 \\ 0 & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -\sin\theta & 0 & 0 \\ 0 & -\sin\theta & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\frac{d\mathbf{y}}{d\theta} = \frac{d}{d\theta} \mathbf{sink}\theta = \frac{d}{d\theta} \begin{pmatrix} \sin\theta & 0 & 0 \\ 0 & \sin\theta & 0 \\ 0 & 0 & \theta \end{pmatrix} = \begin{pmatrix} \cos\theta & 0 & 0 \\ 0 & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{cosk}\theta$$

all of these three methods derivatives are equivalent **(1)** differentiate the basis exponential **expk** itself (same as differentiating the complex exponential), **(2)** differentiate the rotation matrix (which is equivalent to **expk**), and **(3)** differentiate the components of **expk**, **cosk** and **sink**

$$\begin{aligned}
 (1) \quad \frac{d}{d\theta} e^{\theta \hat{k} \times} &= (\hat{k} \times) e^{\theta \hat{k} \times} = (\hat{k} \times) (\cos \theta + (\sin \theta) \hat{k} \times) = \cos \theta (\hat{k} \times) + (\sin \theta) (\hat{k} \times)^2 \\
 &= \begin{pmatrix} \cos \theta & 0 & 0 \\ 0 & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} \sin \theta & 0 & 0 \\ 0 & \sin \theta & 0 \\ 0 & 0 & \theta \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
 &= \begin{pmatrix} -\sin \theta & -\cos \theta & 0 \\ \cos \theta & -\sin \theta & 0 \\ 0 & 0 & 0 \end{pmatrix}
 \end{aligned}$$

$$(2) \quad \frac{d}{d\theta} e^{\theta \hat{k} \times} = \frac{d}{d\theta} \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -\sin \theta & -\cos \theta & 0 \\ \cos \theta & -\sin \theta & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{aligned}
 (3) \quad \frac{d}{d\theta} e^{\theta \hat{k} \times} &= \frac{d}{d\theta} \begin{pmatrix} \cos \theta & 0 & 0 \\ 0 & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{d}{d\theta} \begin{pmatrix} \sin \theta & 0 & 0 \\ 0 & \sin \theta & 0 \\ 0 & 0 & \theta \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -\sin \theta & 0 & 0 \\ 0 & -\sin \theta & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} \cos \theta & 0 & 0 \\ 0 & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
 &= \begin{pmatrix} -\sin \theta & -\cos \theta & 0 \\ \cos \theta & -\sin \theta & 0 \\ 0 & 0 & 0 \end{pmatrix}
 \end{aligned}$$

To show that the Cauchy-Riemann equations hold and are legitimate in the matrix basis, it suffices to show that these derivatives can be obtained with the operator

$$\frac{d}{d\theta} = \frac{d\mathbf{x}}{d\theta} \frac{\partial}{\partial \mathbf{x}} + \frac{d\mathbf{y}}{d\theta} \frac{\partial}{\partial \mathbf{y}}$$

So applying it to the base circle **expk**

$$\frac{d}{d\theta} e^{\theta \hat{k} \times} = \frac{d\mathbf{x}}{d\theta} \frac{\partial}{\partial \mathbf{x}} (\mathbf{x} + \mathbf{y} \hat{k} \times) + \frac{d\mathbf{y}}{d\theta} \frac{\partial}{\partial \mathbf{y}} (\mathbf{x} + \mathbf{y} \hat{k} \times)$$

so

$$\frac{d}{d\theta} e^{\theta \hat{k} \times} = \frac{d\mathbf{x}}{d\theta} \frac{\partial}{\partial \mathbf{x}} (\mathbf{x} + \mathbf{y} \hat{k} \times) + \frac{d\mathbf{y}}{d\theta} \frac{\partial}{\partial \mathbf{y}} (\mathbf{x} + \mathbf{y} \hat{k} \times) = \frac{d\mathbf{x}}{d\theta} \frac{\partial}{\partial \mathbf{x}} (\mathbf{x}) + \frac{d\mathbf{y}}{d\theta} \frac{\partial}{\partial \mathbf{y}} (\mathbf{y} \hat{k} \times) = \frac{d\mathbf{x}}{d\theta} + \frac{d\mathbf{y}}{d\theta} \hat{k} \times$$

that is

$$\frac{d}{d\theta} e^{\theta \hat{k} \times} = \frac{d\mathbf{x}}{d\theta} \frac{\partial}{\partial \mathbf{x}} \begin{pmatrix} x & -y & 0 \\ y & x & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{d\mathbf{y}}{d\theta} \frac{\partial}{\partial \mathbf{y}} \begin{pmatrix} x & -y & 0 \\ y & x & 0 \\ 0 & 0 & 1 \end{pmatrix} = \frac{d\mathbf{x}}{d\theta} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{d\mathbf{y}}{d\theta} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and with

$$\frac{d\mathbf{x}}{d\theta} = \frac{d}{d\theta} \mathbf{cosk}\theta = \frac{d}{d\theta} \begin{pmatrix} \cos\theta & 0 & 0 \\ 0 & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -\sin\theta & 0 & 0 \\ 0 & -\sin\theta & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\frac{d\mathbf{y}}{d\theta} = \frac{d}{d\theta} \mathbf{sink}\theta = \frac{d}{d\theta} \begin{pmatrix} \sin\theta & 0 & 0 \\ 0 & \sin\theta & 0 \\ 0 & 0 & \theta \end{pmatrix} = \begin{pmatrix} \cos\theta & 0 & 0 \\ 0 & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

the total derivative is

$$\frac{d}{d\theta} e^{\theta \hat{k} \times} = \frac{d\mathbf{x}}{d\theta} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{d\mathbf{y}}{d\theta} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -\sin\theta & 0 & 0 \\ 0 & -\sin\theta & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} \cos\theta & 0 & 0 \\ 0 & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -\sin\theta & -\cos\theta & 0 \\ \cos\theta & -\sin\theta & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

which is identical to that previously. So it all holds good similarly in the matrix basis of the diadically expanded operators, specifically here for the Cauchy-Riemann equations and derivatives of complex plane (vector R^3 with rotation axis latent) functions of $i = \hat{k} \times$.

3. Cauchy-Riemann equations for a generalized circle with a single overtone

The simplest of the non-trivial generalized circles is that with a single overtone, that of the first above the base frequency. The generalized circle with the first overtones as a function of **expk** is

$$f(e^{\theta \hat{k} \times}) = f(0) + \frac{1}{1!} f'(0) e^{\theta \hat{k} \times} + \frac{1}{2!} f''(0) e^{2\theta \hat{k} \times} = u(x, y) + i v(x, y)$$

and expanding in terms of **cosk** and **sink**

$$\begin{aligned} f(e^{\theta \hat{k} \times}) &= f(0) + \frac{1}{1!} f'(0) (\mathbf{cosk}\theta + \mathbf{sink}\theta \hat{k} \times) + \frac{1}{2!} f''(0) (\mathbf{cosk}2\theta + \mathbf{sink}2\theta \hat{k} \times) \\ &= (f(0) + \frac{1}{1!} f'(0) \mathbf{cosk}\theta + \frac{1}{2!} f''(0) \mathbf{cosk}2\theta) + (\frac{1}{1!} f'(0) \mathbf{sink}\theta + \frac{1}{2!} f''(0) \mathbf{sink}2\theta) \hat{k} \times \\ &= \mathbf{u}(x, y) + \mathbf{v}(x, y) \hat{k} \times \end{aligned}$$

so the even and odd conjugate operators, **u** and **v**, are

$$\mathbf{u}(x, y) = f(0) + \frac{1}{1!} f'(0) \mathbf{cosk}\theta + \frac{1}{2!} f''(0) \mathbf{cosk}2\theta$$

$$\mathbf{v}(x, y) = \frac{1}{1!} f'(0) \mathbf{sink}\theta + \frac{1}{2!} f''(0) \mathbf{sink}2\theta$$

Applying the operator to the zero phase standard basis

$$\begin{aligned} f(e^{\theta \hat{k} \times}) \hat{i} &= f(0) + \frac{1}{1!} f'(0) (\mathbf{cosk}\theta + \mathbf{sink}\theta \hat{k} \times) + \frac{1}{2!} f''(0) (\mathbf{cosk}2\theta + \mathbf{sink}2\theta \hat{k} \times) \\ &= (f(0) + \frac{1}{1!} f'(0) \mathbf{cosk}\theta + \frac{1}{2!} f''(0) \mathbf{cosk}2\theta) \hat{i} + (\frac{1}{1!} f'(0) \mathbf{sink}\theta + \frac{1}{2!} f''(0) \mathbf{sink}2\theta) \hat{k} \times \hat{i} \\ &= (f(0) + \frac{1}{1!} f'(0) \mathbf{cosk}\theta + \frac{1}{2!} f''(0) \mathbf{cosk}2\theta) \hat{i} + (\frac{1}{1!} f'(0) \mathbf{sink}\theta + \frac{1}{2!} f''(0) \mathbf{sink}2\theta) \hat{j} \\ &= \hat{i} (f(0) + \frac{1}{1!} f'(0) \cos\theta + \frac{1}{2!} f''(0) \cos 2\theta) + \hat{j} (\frac{1}{1!} f'(0) \sin\theta + \frac{1}{2!} f''(0) \sin 2\theta) \end{aligned}$$

After operating on the i -hat component on the standard basis, the even and odd conjugate functions are

$$\begin{aligned} \mathbf{u}(x, y) \hat{i} &= \left(f(0) + \frac{1}{1!} f'(0) \cos \theta + \frac{1}{2!} f''(0) \cos 2\theta \right) \hat{i} = \hat{i} \left(f(0) + \frac{1}{1!} f'(0) \cos \theta + \frac{1}{2!} f''(0) \cos 2\theta \right) \\ \mathbf{v}(x, y) (\hat{k} \times) \hat{i} &= \left(\frac{1}{1!} f'(0) \sin \theta + \frac{1}{2!} f''(0) \sin 2\theta \right) \hat{j} = \hat{j} \left(\frac{1}{1!} f'(0) \sin \theta + \frac{1}{2!} f''(0) \sin 2\theta \right) \end{aligned}$$

which are the first terms of cosine and sine Fourier series. Differentiating

$$f(e^{\theta \hat{k} \times}) \hat{i} = \hat{i} \left(f(0) + \frac{1}{1!} f'(0) \cos \theta + \frac{1}{2!} f''(0) \cos 2\theta \right) + \hat{j} \left(\frac{1}{1!} f'(0) \sin \theta + \frac{1}{2!} f''(0) \sin 2\theta \right)$$

results in

$$\frac{d}{d\theta} f(e^{\theta \hat{k} \times}) \hat{i} = \hat{i} \left(-\frac{1}{1!} f'(0) \sin \theta - \frac{1}{1!} f''(0) \sin 2\theta \right) + \hat{j} \left(\frac{1}{1!} f'(0) \cos \theta + \frac{1}{1!} f''(0) \cos 2\theta \right)$$

where the i -hat and j -hat scalar components are

$$(1) \ u(x, y) = f(0) + \frac{1}{1!} f'(0) \cos \theta + \frac{1}{2!} f''(0) \cos 2\theta \quad \text{and} \quad (2) \ v(x, y) = \frac{1}{1!} f'(0) \sin \theta + \frac{1}{2!} f''(0) \sin 2\theta$$

The relevant derivatives are

$$\begin{aligned} u(x, y) &= f(0) + \frac{1}{1!} f'(0) \cos \theta + \frac{1}{2!} f''(0) \cos 2\theta & v(x, y) &= \frac{1}{1!} f'(0) \sin \theta + \frac{1}{2!} f''(0) \sin 2\theta \\ \frac{du}{d\theta} &= -\frac{1}{1!} f'(0) \sin \theta - \frac{1}{1!} f''(0) \sin 2\theta & \frac{dv}{d\theta} &= \frac{1}{1!} f'(0) \cos \theta + \frac{1}{1!} f''(0) \cos 2\theta \\ u(x, y) &= f(0) + \frac{1}{1!} f'(0)x + \frac{1}{2!} f''(0) \frac{(x+iy)^2 + (x-iy)^2}{2} & v(x, y) &= \frac{1}{1!} f'(0)y + \frac{1}{2!} f''(0) \frac{(x+iy)^2 - (x-iy)^2}{i2} \\ \frac{\partial u}{\partial x} &= f'(0) + f''(0)x & \frac{\partial v}{\partial y} &= f'(0) + f''(0)x \\ \frac{\partial u}{\partial y} &= i \frac{1}{2!} f''(0) \frac{2(x+iy) - 2(x-iy)}{2} = -f''(0)y & \frac{\partial v}{\partial x} &= \frac{1}{2!} f''(0) \frac{2(x+iy) - 2(x-iy)}{i2} = f''(0)y \end{aligned}$$

which demonstrate the Cauchy-Riemann equations hold good for that of the base circle and first overtone,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = f'(0) + f''(0)x \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = f''(0)y$$

and so the generalized circle with first overtone is analytic, $u_x = v_y$ and $v_x = -u_y$. With increasing the sophistication from the base circle to that with the first overtone, the total derivative is increased in sophistication from two terms to four terms

$$\frac{d}{d\theta} = \frac{dx}{d\theta} \frac{\partial}{\partial x} + \frac{dy}{d\theta} \frac{\partial}{\partial y} \quad \text{to} \quad \frac{d}{d\theta} = \frac{dx}{d\theta} \frac{\partial u}{\partial x} \frac{\partial}{\partial u} + \frac{dy}{d\theta} \frac{\partial u}{\partial y} \frac{\partial}{\partial u} + \frac{dx}{d\theta} \frac{\partial v}{\partial x} \frac{\partial}{\partial v} + \frac{dy}{d\theta} \frac{\partial v}{\partial y} \frac{\partial}{\partial v}$$

So for

$$f(e^{\theta \hat{k} \times}) = f(0) + \frac{1}{1!} f'(0) e^{\theta \hat{k} \times} + \frac{1}{2!} f''(0) e^{2\theta \hat{k} \times} = u(x, y) + i v(x, y)$$

$$\frac{d}{d\theta} f(e^{\theta \hat{k} \times}) \hat{i} = \hat{i} \left(-\frac{1}{11} f'(0) \sin \theta - \frac{1}{11} f''(0) \sin 2\theta \right) + \hat{j} \left(\frac{1}{11} f'(0) \cos \theta + \frac{1}{11} f''(0) \cos 2\theta \right)$$

applying the total derivative that recognizes each sub-function to the generalized function $f(\vec{r})$

$$\begin{aligned} \frac{df(\vec{r})}{d\theta} &= \left(\frac{d}{d\theta} = \frac{dx}{d\theta} \frac{\partial u}{\partial x} \frac{\partial}{\partial u} + \frac{dy}{d\theta} \frac{\partial u}{\partial y} \frac{\partial}{\partial u} + \frac{dx}{d\theta} \frac{\partial v}{\partial x} \frac{\partial}{\partial v} + \frac{dy}{d\theta} \frac{\partial v}{\partial y} \frac{\partial}{\partial v} \right) f(\vec{r}) \\ &= \left(\frac{dx}{d\theta} \frac{\partial u}{\partial x} + \frac{dy}{d\theta} \frac{\partial u}{\partial y} \right) \frac{\partial f(\vec{r})}{\partial u} + \left(\frac{dx}{d\theta} \frac{\partial v}{\partial x} + \frac{dy}{d\theta} \frac{\partial v}{\partial y} \right) \frac{\partial f(\vec{r})}{\partial v} \quad \text{where } \frac{dx}{d\theta} = -\sin \theta = -y \text{ and } \frac{dy}{d\theta} = \cos \theta = x \\ &= \left(-y \frac{\partial u}{\partial x} + x \frac{\partial u}{\partial y} \right) \frac{\partial f(\vec{r})}{\partial u} + \left(-y \frac{\partial v}{\partial x} + x \frac{\partial v}{\partial y} \right) \frac{\partial f(\vec{r})}{\partial v} \end{aligned}$$

$$\begin{aligned} \frac{df(\vec{r})}{d\theta} &= (-y(f'(0) + f''(0)x) + x(-f''(0)y)) \frac{\partial f(\vec{r})}{\partial u} + (-y(f''(0)y) + x(f'(0) + f''(0)x)) \frac{\partial f(\vec{r})}{\partial v} \\ &= (-f'(0)y - f''(0)xy - f''(0)yx) \frac{\partial f(\vec{r})}{\partial u} + (-f''(0)y^2 + f'(0)x + f''(0)x^2) \frac{\partial f(\vec{r})}{\partial v} \\ &= (-f'(0)y - f''(0)2xy) \frac{\partial f(\vec{r})}{\partial u} + (f'(0)x + f''(0)(x^2 - y^2)) \frac{\partial f(\vec{r})}{\partial v} \\ &= (-f'(0)y - f''(0)2xy) (\hat{i}1 + \hat{j}0) + (f'(0)x + f''(0)(x^2 - y^2)) (\hat{i}0 + \hat{j}1) \\ &= \hat{i}(-f'(0)y - f''(0)2xy) + \hat{j}(f'(0)x + f''(0)(x^2 - y^2)) \end{aligned}$$

after inserting the Cauchy-Riemann partials; and finally substituting in the spatial even cosine and odd conjugate sine of the base circle $x = \cos \theta$ and $y = \sin \theta$

$$\begin{aligned} \frac{df(\vec{r})}{d\theta} &= \hat{i}(-f'(0)y - f''(0)2xy) + \hat{j}(f'(0)x + f''(0)(x^2 - y^2)) \\ &= \hat{i}(-f'(0)\sin \theta - f''(0)2\cos \theta \sin \theta) + \hat{j}(f'(0)\cos \theta + f''(0)(\cos^2 \theta - \sin^2 \theta)) \\ &= \hat{i}(-f'(0)\sin \theta - f''(0)\sin 2\theta) + \hat{j}(f'(0)\cos \theta + f''(0)\cos 2\theta) \end{aligned}$$

i.e.,

$$\frac{d}{d\theta} f(e^{\theta \hat{k} \times}) \hat{i} = \hat{i}(-f'(0)\sin \theta - f''(0)\sin 2\theta) + \hat{j}(f'(0)\cos \theta + f''(0)\cos 2\theta)$$

which was to be demonstrated,

$$\frac{df(\vec{r})}{d\theta} = \hat{i} \frac{d\mathbf{u}}{d\theta} + \hat{j} \frac{d\mathbf{v}}{d\theta}$$

It all works as it should. The Cauchy-Riemann equations hold good, the partials are accurate and combine correctly into the total derivative of the vector function. Now to take the derivatives to second order to demonstrate that a generalized circle satisfies the Laplace equation, and are hence harmonic. Here for the simplest of the generalized circle, a base circle with a single overtone, the partial derivatives for the Cauchy-Riemann equations were

$$\frac{\partial u}{\partial x} = \frac{1}{1!} f'(0) + \frac{1}{1!} f''(0)x$$

$$\frac{\partial u}{\partial y} = i \frac{1}{2!} f''(0) \frac{2(x+iy) - 2(x-iy)}{2} = -\frac{1}{1!} f''(0)y$$

$$\frac{\partial v}{\partial y} = \frac{1}{1!} f'(0) + \frac{1}{1!} f''(0)x$$

$$\frac{\partial v}{\partial x} = \frac{1}{2!} f''(0) \frac{2(x+iy) - 2(x-iy)}{i2} = \frac{1}{1!} f''(0)y$$

the second order partials are

$$\frac{\partial^2 u}{\partial x^2} = f''(0) \text{ and } \frac{\partial^2 u}{\partial y^2} = -f''(0) \quad \text{so even } u \text{ satisfies the Laplace equation } \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(0) - f''(0) = 0 \text{ i.e., } u_{xx} + u_{yy} = 0$$

$$\frac{\partial^2 v}{\partial y^2} = 0 \text{ and } \frac{\partial^2 v}{\partial x^2} = 0 \quad \text{so odd } v \text{ trivially satisfies the Laplace equation } \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial x^2} = 0 + 0 = 0$$

so that the generalized circle itself also satisfies Laplace equation, and is hence harmonic

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) f(\vec{r}) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) u(x, y) + i \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) v(x, y) = 0 + i0$$

that is,

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

Note also the cross-second order derivative $\frac{\partial^2 v}{\partial y \partial x} = f''(0)$ so $\frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial y \partial x} = f''(0)$. It all works in matrix bases as well.

This is omitted so as not to become too many additional pages (now at 132 when it was only supposed to be about half a dozen for $i = \hat{k} \times \dots$. I could convert over all of Witteraker and Watson [with very large expansions in theory] and the full standard texts to the new theory although this would be the size of telephone books... if someone else wants to, that would be great; it is a number one opportunity for new research). Now for one more, a generalized circle of the base circle with the first two overtones where the second order partial derivative in v is non-trivial.

4. Cauchy-Riemann equations for a generalized circle with the first two overtones

The generalized circle with the first two overtones as a function of **expk**

$$f(e^{\theta \hat{k} \times}) = f(0) + \frac{1}{1!} f'(0) e^{\theta \hat{k} \times} + \frac{1}{2!} f''(0) e^{2\theta \hat{k} \times} + \frac{1}{3!} f'''(0) e^{3\theta \hat{k} \times} \\ = \mathbf{u}(x, y) + \mathbf{v}(x, y) \hat{k} \times$$

and expanding in terms of **cosk** and **sink**

$$\begin{aligned}
f(e^{\theta \hat{k} \times}) &= f(0) + \frac{1}{1!} f'(0) (\cos \theta + \sin \theta \hat{k} \times) + \frac{1}{2!} f''(0) (\cos 2\theta + \sin 2\theta \hat{k} \times) + \frac{1}{3!} f'''(0) (\cos 3\theta + \sin 3\theta \hat{k} \times) \\
&= \left(f(0) + \frac{1}{1!} f'(0) \cos \theta + \frac{1}{2!} f''(0) \cos 2\theta + \frac{1}{3!} f'''(0) \cos 3\theta \right) + \left(\frac{1}{1!} f'(0) \sin \theta + \frac{1}{2!} f''(0) \sin 2\theta + \frac{1}{3!} f'''(0) \sin 3\theta \right) \hat{k} \times \\
&= \mathbf{u}(x, y) + \mathbf{v}(x, y) \hat{k} \times
\end{aligned}$$

so the even and odd conjugate operators, $\mathbf{u}$ and $\mathbf{v}$, are

$$\mathbf{u}(x, y) = f(0) + \frac{1}{1!} f'(0) \cos \theta + \frac{1}{2!} f''(0) \cos 2\theta + \frac{1}{3!} f'''(0) \cos 3\theta$$

$$\mathbf{v}(x, y) = \frac{1}{1!} f'(0) \sin \theta + \frac{1}{2!} f''(0) \sin 2\theta + \frac{1}{3!} f'''(0) \sin 3\theta$$

Applying the operator to the zero phase standard basis

$$\begin{aligned}
f(e^{\theta \hat{k} \times}) \hat{i} &= f(0) + \frac{1}{1!} f'(0) (\cos \theta + \sin \theta \hat{k} \times) + \frac{1}{2!} f''(0) (\cos 2\theta + \sin 2\theta \hat{k} \times) \\
&= \left(f(0) + \frac{1}{1!} f'(0) \cos \theta + \frac{1}{2!} f''(0) \cos 2\theta \right) \hat{i} + \left(\frac{1}{1!} f'(0) \sin \theta + \frac{1}{2!} f''(0) \sin 2\theta \right) \hat{k} \times \hat{i} \\
&= \left(f(0) + \frac{1}{1!} f'(0) \cos \theta + \frac{1}{2!} f''(0) \cos 2\theta \right) \hat{i} + \left(\frac{1}{1!} f'(0) \sin \theta + \frac{1}{2!} f''(0) \sin 2\theta \right) \hat{j} \\
&= \hat{i} \left(f(0) + \frac{1}{1!} f'(0) \cos \theta + \frac{1}{2!} f''(0) \cos 2\theta \right) + \hat{j} \left(\frac{1}{1!} f'(0) \sin \theta + \frac{1}{2!} f''(0) \sin 2\theta \right)
\end{aligned}$$

After operating on the i -hat component on the standard basis, the even and conjugate odd functions are

$$\begin{aligned}
\mathbf{u}(x, y) \hat{i} &= \left(f(0) + \frac{1}{1!} f'(0) \cos \theta + \frac{1}{2!} f''(0) \cos 2\theta + \frac{1}{3!} f'''(0) \cos 3\theta \right) \hat{i} = \hat{i} \left(f(0) + \frac{1}{1!} f'(0) \cos \theta + \frac{1}{2!} f''(0) \cos 2\theta + \frac{1}{3!} f'''(0) \cos 3\theta \right) \\
\mathbf{v}(x, y) (\hat{k} \times) \hat{i} &= \left(\frac{1}{1!} f'(0) \sin \theta + \frac{1}{2!} f''(0) \sin 2\theta + \frac{1}{3!} f'''(0) \sin 3\theta \right) \hat{j} = \hat{j} \left(\frac{1}{1!} f'(0) \sin \theta + \frac{1}{2!} f''(0) \sin 2\theta + \frac{1}{3!} f'''(0) \sin 3\theta \right)
\end{aligned}$$

which are the first terms of cosine and sine Fourier series. Differentiating

$$f(e^{\theta \hat{k} \times}) \hat{i} = \hat{i} \left(f(0) + \frac{1}{1!} f'(0) \cos \theta + \frac{1}{2!} f''(0) \cos 2\theta + \frac{1}{3!} f'''(0) \cos 3\theta \right) + \hat{j} \left(\frac{1}{1!} f'(0) \sin \theta + \frac{1}{2!} f''(0) \sin 2\theta + \frac{1}{3!} f'''(0) \sin 3\theta \right)$$

results in

$$\frac{d}{d\theta} f(e^{\theta \hat{k} \times}) \hat{i} = \hat{i} \left(-f'(0) \sin \theta - \frac{1}{1!} f''(0) \sin 2\theta - \frac{1}{2!} f'''(0) \sin 3\theta \right) + \hat{j} \left(f'(0) \cos \theta + \frac{1}{1!} f''(0) \cos 2\theta + \frac{1}{2!} f'''(0) \cos 3\theta \right)$$

where the i -hat and j -hat scalar components are, for $u(x, y)$

- $$\begin{aligned}
u(x, y) &= f(0) + \frac{1}{1!} f'(0) \cos \theta + \frac{1}{2!} f''(0) \cos 2\theta + \frac{1}{3!} f'''(0) \cos 3\theta \\
\frac{du}{d\theta} &= -\frac{1}{1!} f'(0) \sin \theta - \frac{1}{1!} f''(0) \sin 2\theta - \frac{1}{3!} f'''(0) \sin 3\theta \\
u(x, y) &= f(0) + \frac{1}{1!} f'(0)x + \frac{1}{2!} f''(0) \frac{(x+iy)^2 + (x-iy)^2}{2} + \frac{1}{3!} f'''(0) \frac{(x+iy)^3 + (x-iy)^3}{2} \\
\frac{\partial u}{\partial x} &= f'(0) + f''(0)x + \frac{1}{2} f'''(0) \frac{(x+iy)^2 + (x-iy)^2}{2} = f'(0) + f''(0)x + \frac{1}{2} f'''(0)(x^2 - y^2) \\
\frac{\partial u}{\partial y} &= \frac{1}{2!} f''(0) \frac{2i(x+iy) - 2i(x-iy)}{2} + \frac{1}{3!} f'''(0) \frac{3i(x+iy)^2 - 3i(x-iy)^2}{2} = -f''(0)y - \frac{1}{2!} f'''(0)2xy
\end{aligned}$$

and for $v(x, y)$

$$\begin{aligned}
\bullet \quad v(x, y) &= \frac{1}{1!} f'(0) \sin \theta + \frac{1}{2!} f''(0) \sin 2\theta + \frac{1}{3!} f'''(0) \sin 3\theta \\
\frac{dv}{d\theta} &= \frac{1}{1!} f'(0) \cos \theta + \frac{1}{1!} f''(0) \cos 2\theta + \frac{1}{2!} f'''(0) \cos 3\theta \\
v(x, y) &= \frac{1}{1!} f'(0)y + \frac{1}{2!} f''(0) \frac{(x+iy)^2 - (x-iy)^2}{i2} + \frac{1}{3!} f'''(0) \frac{(x+iy)^3 - (x-iy)^3}{i2} \\
\frac{\partial v}{\partial y} &= f'(0) + f''(0)x + \frac{1}{2!} f'''(0) \frac{(x+iy)^2 + (x-iy)^2}{2} = f'(0) + f''(0)x + \frac{1}{2!} f'''(0)(x^2 - y^2) \\
\frac{\partial v}{\partial x} &= \frac{1}{2!} f''(0) \frac{2(x+iy) - 2(x-iy)}{i2} + \frac{1}{3!} f'''(0) \frac{3(x+iy)^2 - 3(x-iy)^2}{i2} = f''(0)y + \frac{1}{2!} f'''(0)2xy
\end{aligned}$$

Hence, the Cauchy-Riemann equations $u_x = v_y$ and $v_x = -u_y$ hold good

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = f'(0) + f''(0)x + \frac{1}{2} f'''(0)(x^2 - y^2) \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = f''(0)y + f'''(0)xy$$

For Laplace's equation, the second order partial derivatives are

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} &= f''(0) + f'''(0)x & \text{and} & \quad \frac{\partial^2 v}{\partial x \partial y} = -\frac{\partial^2 u}{\partial y^2} = f''(0) + f'''(0)x \\
\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 v}{\partial y^2} &= -f'''(0)y & \text{and} & \quad \frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 u}{\partial x \partial y} = f'''(0)y
\end{aligned}$$

so $u_{xx} + u_{yy} = 0$ and $v_{xx} + v_{yy} = 0$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = (f''(0) + f'''(0)x) - (f''(0) + f'''(0)x) = 0 \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = -f'''(0)y + f'''(0)y = 0$$

Therefore,

$$f_{xx}(e^{\theta \hat{k} \times}) + f_{yy}(e^{\theta \hat{k} \times}) = (u_{xx} + iv_{xx}) + i(u_{yy} + v_{yy}) = (u_{xx} + u_{yy}) + i(v_{xx} + v_{yy}) = 0$$

the generalized circle satisfies Laplace's equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

noting that the second order partial derivatives can also be written as

$$\begin{pmatrix} \frac{\partial^2}{\partial x^2} & -\frac{\partial^2}{\partial x \partial y} \\ -\frac{\partial^2}{\partial x \partial y} & \frac{\partial^2}{\partial y^2} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \frac{\partial^2}{\partial y^2} & \frac{\partial^2}{\partial x \partial y} \\ \frac{\partial^2}{\partial x \partial y} & \frac{\partial^2}{\partial x^2} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

although the operators are singular. Also note that the partial derivatives can be taken to higher orders than the Laplacian (until zeroing and becoming null) to the number of overtones in the generalized circle (the number of terms in the power series of **expk**).

To demonstrate that the partial derivatives are legitimate, here for two overtones, it is shown now that the total derivative of the generalized circle can be obtained by the chained partial derivatives for the parameterized spatial coordinates $x = \cos \theta$ and $y = \sin \theta$

$$\frac{d}{d\theta} = \frac{dx}{d\theta} \frac{\partial u}{\partial x} \frac{\partial}{\partial u} + \frac{dy}{d\theta} \frac{\partial u}{\partial y} \frac{\partial}{\partial u} + \frac{dx}{d\theta} \frac{\partial v}{\partial x} \frac{\partial}{\partial v} + \frac{dy}{d\theta} \frac{\partial v}{\partial y} \frac{\partial}{\partial v} = \left(-y \frac{\partial u}{\partial x} + x \frac{\partial u}{\partial y} \right) \frac{\partial}{\partial u} + \left(-y \frac{\partial v}{\partial x} + x \frac{\partial v}{\partial y} \right) \frac{\partial}{\partial v}$$

Applying the operator for to the generalized circle $f(e^{\theta \hat{k} \times}) = f(0) + \frac{1}{1!} f'(0) e^{\theta \hat{k} \times} + \frac{1}{2!} f''(0) e^{2\theta \hat{k} \times} = u(x, y) + i v(x, y)$

$$\begin{aligned} \frac{df(\vec{r})}{d\theta} &= \left(\frac{dx}{d\theta} \frac{\partial u}{\partial x} \frac{\partial}{\partial u} + \frac{dy}{d\theta} \frac{\partial u}{\partial y} \frac{\partial}{\partial u} + \frac{dx}{d\theta} \frac{\partial v}{\partial x} \frac{\partial}{\partial v} + \frac{dy}{d\theta} \frac{\partial v}{\partial y} \frac{\partial}{\partial v} \right) f(\vec{r}) \\ &= \left(-y \frac{\partial u}{\partial x} + x \frac{\partial u}{\partial y} \right) \frac{\partial f(\vec{r})}{\partial u} + \left(-y \frac{\partial v}{\partial x} + x \frac{\partial v}{\partial y} \right) \frac{\partial f(\vec{r})}{\partial v} \\ &= \left(-y \frac{\partial u}{\partial x} + x \frac{\partial u}{\partial y} \right) (\hat{i}1 + \hat{j}0) + \left(-y \frac{\partial v}{\partial x} + x \frac{\partial v}{\partial y} \right) (\hat{i}0 + \hat{j}1) \\ &= \left(-y(f'(0) + f''(0)x + \frac{1}{2} f'''(0)(x^2 - y^2)) + x(-f''(0)y - \frac{1}{2!} f'''(0)2xy) \right) (\hat{i}1 + \hat{j}0) \\ &\quad + \left(-y(f''(0)y + \frac{1}{2!} f'''(0)2xy) + x(f'(0) + f''(0)x + \frac{1}{2!} f'''(0)(x^2 - y^2)) \right) (\hat{i}0 + \hat{j}1) \\ &= \left((-f'(0)y - f''(0)xy - \frac{1}{2} f'''(0)(x^2 y - y^3)) + (-f''(0)xy - \frac{1}{2!} f'''(0)2x^2 y) \right) (\hat{i}1 + \hat{j}0) \\ &\quad + \left((-f''(0)y^2 - \frac{1}{2!} f'''(0)2xy^2) + (f'(0)x + f''(0)x^2 + \frac{1}{2!} f'''(0)(x^3 - xy^2)) \right) (\hat{i}0 + \hat{j}1) \\ &= \left(-f'(0)y - f''(0)2xy - \frac{1}{2} f'''(0)(3x^2 y - y^3) \right) (\hat{i}1 + \hat{j}0) \\ &\quad + \left(f'(0)x + f''(0)(x^2 - y^2) + \frac{1}{2!} f'''(0)(x^3 - 3xy^2) \right) (\hat{i}0 + \hat{j}1) \end{aligned}$$

and finally with $x = \cos \theta$, $y = \sin \theta$, $2xy = \sin 2\theta$, $x^2 - y^2 = \cos 2\theta$, $x^3 - 3xy^2 = \cos 3\theta$, and $3x^2 y - y^3 = \sin 3\theta$

$$\frac{df(\vec{r})}{d\theta} = \hat{i} \left(-f'(0) \sin \theta - f''(0) 2 \sin 2\theta - \frac{1}{2} f'''(0) \sin 3\theta \right) + \hat{j} \left(f'(0) \cos \theta + f''(0) \cos 2\theta + \frac{1}{2!} f'''(0) \cos 3\theta \right)$$

which is the correct total derivate as that obtained without chaining the partial derivatives, demonstrating that the partial derivatives in x and y for the Cauchy-Riemann and Laplace equations are correct, albeit more of academic value here. It is more of value to demonstrate that they are not themselves transfer functions, but rather the inner workings of transfer functions (the inner workings of transfer functions such as power series compositions of **expk** for instance).